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This report summarizes the results obtained during the second year of the project. The goal of the project as a whole is the investigation of fundamental bounds on the maximal achievable precision of aiming of dynamical systems with random perturbations and application of these bounds to control of space structures. To this end, during the second year of the project the following results have been obtained: it has been shown that linear systems with small additive noise can be pointed with any desired accuracy by *output* feedback if and only if the system is invertable and minimum phase in an approximate sense; when the measurements noise is present, the maximal achievable precision of aiming is bounded, even if the conditions mentioned above are satisfied; thus, the measurement noise has a more severe effect on the pointabilty of dynamical system than the input noise. In addition, the problem of residence probability control has been investigated and its relation to the problem of residence time control has been analyzed.

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Table of Contents

SECTION I
INTRODUCTION

1. OBJECTIVES AND STATUS OF THE RESEARCH

This annual report describes the results obtained during the second year of the project directed towards the analysis of fundamental bounds on the maximal achievable precision of aiming of dynamical systems with random disturbances and application of these bounds to control of space structures. The specific goals of the second year of the study were as follows: Conduct research into and uncover the fundamental properties of the following topics:

1. State and output aiming controllers with output feedback: analysis and design.

2. Aiming controllers with noisy measurement: fundamental tradeoffs.

3. Residence proability control vs. residence time control: analysis and synthesis.

These goals have been achieved and the results are reported in Sections II-III below. The summary of the results is as follows:

Section II is devoted to analysis of residence time control processes using the dynamic output feedback. The main conclusions arrived at in this study are:

1. The linear system

$$
\begin{aligned}
dx &= (Ax + Bu)dt + \epsilon C dw \ , \quad x(0) = x_0 \\
y &= Dx \ ,
\end{aligned}
\tag{1.1}
$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$, w is a standard r-dimensional Brownian motion and $0 < \epsilon \ll 1$, with the noiseless output feedback

$$
\begin{aligned}
u &= K\hat{x} \\
\dot{\hat{x}} &= A\hat{x} + Bu + L(z - E\hat{x}) \ , \quad z = Ex \ , \quad z \in R^q \ ,
\end{aligned}
\tag{1.2}
$$

is pointable with any desired residence time if and only if it is invertible and minimum phase in an appropriate sense.

2. If it is not the case, the maximal achievable residence time $T^* < \infty$ for system (1.1), (1.2) coincides with that for system with state feedback, $u = Kx$, if and only if $E(sI - A)^{-1}C$ is left invertible and minimum phase; otherwise, the output controllers give a smaller residence time in comparison with the state space controllers.

3. The residence time of system (1.1) with noisy output feedback

$$
\begin{aligned}
u &= K\hat{x} \\
d\hat{x} &= (A\hat{x} + Bu)dt + L(dz - E\hat{x}dt) \\
dz &= Exdt + \epsilon F dw_1 \ ,
\end{aligned}
\tag{1.3}
$$

where $w_1(t)$ is a q-dimensional standard Brownian motion, is always bounded,

$T^* < \infty$. Thus, the measurement noise has a more severe effect on the residence time than the input noise.

4. The observer gain L that ensures the largest possible residence time in system (1.1), (1.3) coincides with that of the corresponding Kalman filter. Thus, Kalman filter is optimal not only with respect to the standard performance measure, i.e., the mean square estimation error, but also from the point of view of the residence time.

5. The feedback gain K that ensures the largest possible residence time in system (1.1), (1.3) is dependent on the optimal value of L mentioned above. Thus, although the separation principle does not take place, the situation here can be characterized as semi-separation: the optimal observations do not depend on optimal control but the optimal control does depend on optimal observations. As a result, the maximal achievable residence time for controllers derived in this paper is larger than that for LQG-designed systems.

Section III is devoted to the formulation, justification, and solution of the residence ***probability*** control problem and to the introduction and analysis of the notion of (D,T)-stability. Specifically, if

$$\tau_{x_0} = \inf\{t > 0 : \ x(t) \in \partial D | x_0 \in [D_0]\} \ , \tag{1.4}$$

($D \subset R^n$ and $D_0 \subset D$ are open bounded domains with 0 in their interiors) is the

first passage time of the trajectory of (1.1) with

$$u = Kx \ , \tag{1.5}$$

from D, than the problem of aiming control can be reformulated in the two following ways:

Residence time control: Given (1.1) and a pair (D, T), find a feedback law (1.5) and an open set $D_0 \subset D$ such that

$$E[\tau_{x_0}] \geq T \ , \quad \forall \ x_0 \in [D_0] \ . \tag{1.6}$$

Residence proability control: Given (1.1), a pair (D, T) and a constant $0 < p < 1$, find a feedback law (1.5) and an open set $D_0 \subset D$ such that

$$\text{Prob}\{\tau_{x_0} > T\} > p \ , \quad \forall \ x_0 \in [D_0] \ . \tag{1.7}$$

The residence time control problem (1.6) and its generalizations has been analyzed during the first year of the project and in Section II of this report.

The residence probability control (1.7) appears to be a stronger reformulation of the aiming control problem than the residence time control. Indeed, since τ_{x_0} is non-negative random variable, the Markov inequality gives:

$$\text{Prob}\{\tau_{x_0} \geq T\} \leq \frac{E[\tau_{x_0}]}{T} \ .$$

Therefore, if $\text{Prob}\{\tau_{x_0} \geq T\} \geq p$, the estimate for $E[\tau_{x_0}]$ follows immediately:

$$E[\tau_{x_0}] \geq pT \ .$$

Moreover, it is possible to show that for any D, D_0, T and $0 < p < 1$ there exists a feedback law (1.5) and $x_0 \in D_0$ such that the closed loop system (1.1), (1.5) has the following property:

$$E[\tau_{x_0}] > T \quad ,$$

$$\text{Prob}\{\tau_{x_0} > T\} < p \quad .$$

This implies that the closed loop system may exhibit a performance as good as desired from the residence time point of view and as bad as desired from the point of view of the residence probability.

These observations justify the problem of residence probability control.

The results obtained in Section III with regard to the residence probability control problem can be summarized as follows:

1. The residence probability control gives a stronger reformulation of the aiming control problem than the residence time control. However, the resulting control problem is also more complex: the performance depends on the size of the initial, "lock in", domain and on the operating period.

2. The (D, T)-stability with probability p is a useful tool for characterization of the performance of stochastic systems with no equilibrium points. The performance in terms of (D, T)-stability may be contradictory to the performance in terms of LQG criterion.

3. The residence probability of a controlled linear system with additive white

noise an be modified in any desired manner, for instance, made as close to 1 as desired, if and only if the range space of the noise matrix is included in the range space of the control matrix. Otherwise, the achievable residence probability is bounded away from 1 and estimates of this bound are characterized herein (Section III).

2. PROFESSIONAL PERSONNEL ASSOCIATED WITH THE PROJECT

The results outlined above were obtained by a research team that included:

1. Semyon M. Meerkov, Professor, Principal Investigator.

2. Thordur Runolfsson, Post Doctoral Fellow (August 1, 1988 - January 15, 1989). Starting from January 16, 1989, Dr. Runolfsson has accepted a position of an Assistant Professor in the Department of Electrical and Computer Engineering, The Johns Hopkins University, Baltimore, MD 21218.

3. Seungnam Kim, Doctoral Candidate.

3. PAPERS IN THE AREA OF THE PROJECT

[1]. A paper titled "Theory of Residence Time Control by Output Feedback", by S.M. Meerkov and T. Runolfsson has been accepted for presentation at the *28th IEEE CDC*, Tampa, FL, Dec. 1989. This paper comprises Section II of this report.

[2]. A paper, "Theory of Aiming Control for Linear Stochastic Systems", by S. Meerkov and T. Runolfsson has been submitted to the *11th IFAC Congress*, Tallinn, U.S.S.R., Aug. 1990. This paper is included as an Appendix herewith.

[3]. A paper, "Aiming Control: Residence Probability and (D,T)-stability, by S. Kim, S.M. Meerkov and T. Runolfsson, has been prepared for submission to the *29th IEEE CDC*. This paper comprises Section III of this report.

[4]. A paper: "Aiming Control: Design of Residence Probability Controllers" is being prepared for submission to the *29th IEEE CDC*.

[5]. A paper, "Output Residence Time Control" by S.M. Meerkov and T. Runolfsson (reported originally in the Annual Report for 1987-1988) is to appear in the *IEEE Trans. Automat. Contr.*. The galley proofs of this paper are included in the Appendix.

SECTION II

THEORY OF RESIDENCE TIME CONTROL BY OUTPUT FEEDBACK

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Abstract

The problem of residence time control by the observer based output feedback is formulated and solved for the case of linear systems with small additive input noise. Both noiseless and noisy measurements are considered. In the noiseless measurements case, it is shown that the fundamental bounds on the achievable residence time depend on the nonminimum phase zeros of the system. In the noisy measurements case, the achievable residence time is shown to be always bounded, and an estimate of this bound is given. Controller design techniques are presented. The development is based on the asymptotic large deviations theory.

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1. INTRODUCTION

Consider the following Ito stochastic system

$$\begin{aligned} dx &= (Ax + Bu)dt + \epsilon C dw \\ y &= Dx \ , \end{aligned} \tag{1.1}$$

where $x \in \boldsymbol{R}^n, u \in \boldsymbol{R}^m, y \in \boldsymbol{R}^p, w(t)$ is a standard r-dimensional Brownian motion, A, B, C, D are matrices of appropriate dimensionality, and $0 < \epsilon \ll 1$. For a given u, the behavior of (1.1) in a bounded domain $\Psi \subset \boldsymbol{R}^p$ can be characterized by the first passage time [1],

$$\tau^\epsilon(u) = \inf\{t \geq 0 : \ y(t,u) \in \partial\Psi | y(0,u) \in \Psi\} \ ,$$

($\partial\Psi$ is the boundary of Ψ), or by its average value

$$\bar{\tau}^\epsilon(u) = E[\tau^\epsilon(u)] \ .$$

The $\bar{\tau}^\epsilon(u)$ is referred to as the (average) residence time of (1.1) in Ψ.

Assume that control specifications for (1.1) are given in the form of an aiming (pointing) problem: maintain $y(t)$ in a given domain $\Psi \subset \boldsymbol{R}^p$ during a specified time interval $[0, T]$, $T < \infty$. In terms of the average residence time this problem has the form

$$\bar{\tau}^\epsilon(u) \geq T \ . \tag{1.2}$$

Technical examples of this problem can be found in [2].

To accomplish (1.2), the feedback control approach can be utilized. Papers [2] and [3] address this problem under the assumption that all states x are available for control and u is chosen as

$$u = Kx \quad . \tag{1.3}$$

In [2] it was assumed that $D = I$, i.e. the pointing of states has been considered, and in [3] the general case of output aiming has been analyzed. It has been shown that from the point of view of satisfying (1.2), all systems (1.1) can be partitioned into two groups: weakly and strongly residence time controllable. Roughly speaking, (1.1) is weakly residence time controllable (wrt-controllable) if there exists $T^* < \infty$ such that the closed loop system (1.1), (1.3) satisfies (1.2) for $T < T^*$ and some K and does not satisfy (1.2) for $T > T^*$ and any K. System (1.1) is strongly residence time controllable (srt-controllable) if $T^* = \infty$. It has been shown in [2] that (1.1) with $D = I$ is wrt-controllable if and only if (A, B) is stabilizable and srt-controllable if and only if $Im\ C \subseteq Im\ B$. It has been shown in [3] that system (1.1) wrt-controllable in states can, in fact, be srt-controllable in outputs $y \neq x$. In particular, it was shown that a SISO system (1.1) is srt-controllable if and only if all non-minimum phase zeros of $G_s(s) \triangleq D(sI - A)^{-1}B$ coincide with non-minimum phase zeros of $G_n(s) \triangleq D(sI - A)^{-1}C$. This means, of course, that minimum phase plants are pointable with any precision whereas non-minimum phase ones may or may not be, depending on the location of the right half plane zeros of $G_n(s)$.

In the present paper we address problem (1.2) under the assumption that only (measured) outputs are available for control and, therefore, the output feedback has to be utilized. To simplify the problem, we consider here the observer based output feedback, i.e. controllers of the form:

$$\begin{aligned} u &= K\hat{x} \\ \dot{\hat{x}} &= A\hat{x} + Bu + L(z - E\hat{x}) \end{aligned} \tag{1.4}$$

if the measured output,

$$z = Ex \ , \quad z \in \mathbb{R}^q \ , \quad E \in \mathbb{R}^{q\times n} \ ,$$

is noise free, or

$$\begin{aligned} u &= K\hat{x} \\ d\hat{x} &= (A\hat{x} + Bu)dt + L(dz - E\hat{x}dt) \end{aligned} \tag{1.5}$$

if the measured output,

$$dz = Exdt + \epsilon \ Fdw_1 \quad ,$$

is noisy. Here $w_1(t)$ is a q-dimensional standard Brownian motion and $0 < \epsilon \ll 1$. In each case, (1.4) and (1.5), the problem is to choose the pair (K, L) so that (1.2) is satisfied.

To this end, in this paper we derive the following results:

1. System (1.1) with feedback (1.4) is *srt*-controllable if and only if the system is invertible and minimum phase in an appropriate sense.

2. If this is not the case, the maximal achievable residence time T^* for system (1.1), (1.4) coincides with that for system (1.1), (1.3) if and only if $G_{n_1}(s) \triangleq E(sI - A)^{-1}C$ is left invertible and minimum phase; otherwise the output controllers lead to a smaller residence time.

3. System (1.1) with feedback (1.5) is never *srt*-controllable. Thus, the measurement noise has a much more severe effect on the residence time than the input noise.

4. The observer gain L that ensures the largest possible residence time in system (1.1), (1.5) coincides with that of the corresponding Kalman filter. Thus, Kalman filter is optimal not only with respect to the standard performance measure, i.e., the mean square estimation error, but also from the point of view of the residence time.

5. The feedback gain K that ensures the largest possible residence time in system (1.1), (1.5) is dependent on the optimal value of L mentioned above. Thus, although the separation principle does not take place, the situation here can be characterized as semi-separation: the optimal observations do not depend on optimal control but the optimal control does depend on optimal observations. As a result, the maximal achievable residence time for controllers derived in this paper is larger than that for LQG-designed systems.

The remainder of this paper is organized as follows: In Section 2 some mathematical preliminaries are discussed. In Sections 3 - 5 system (1.1) with controllers

(1.4) and (1.5), respectively, is considered and in Section 6 an illustrative example is given. In Section 7 the conclusions are formulated. The proofs are given in the Appendix.

2. PRELIMINARIES

In this section, the notion of logarithmic residence time, i.e., the main tool of asymptotic analysis of (1.1) with (1.3)-(1.5), is introduced and utilized for a precise formulation of problem (1.2).

Consider the linear Ito system

$$\begin{aligned} dx &= Axdt + \epsilon\, Cdw \\ y &= Dx \end{aligned} \tag{2.1}$$

where, as before, $x \in R^n, y \in R^p, w(t)$ is a standard r-dimensional Brownian motion and $0 < \epsilon \ll 1$. Let $\Psi \subset R^p$ be again a bounded domain with the origin in its interior and a smooth boundary $\partial\Psi$. Define

$$\Omega_0 \triangleq \{x \in R^n : \; y = Dx \in \Psi\} \; , \tag{2.2}$$

$$\Omega \triangleq \{x \in R^n : \; De^{At}x \in \Psi, \; t \geq 0\} \; . \tag{2.3}$$

Assume that $x(0) = x_0 \in \Omega_0$ and introduce the first passage time as

$$\tau^\epsilon(x_0) \triangleq \inf\{t \geq 0 : \; y(t, x_0) \in \partial\Psi\} \; , \tag{2.4}$$

where $y(t, x_0)$ is the solution of (2.1). The following theorem was proved in [3].

Theorem 2.1: Suppose A is Hurwitz and (A, C) is disturbable, i.e. rank $[C\ AC \dots A^{n-1}C] = n$. Then uniformly for all x_0 belonging to compact subsets of Ω we have:

$$\lim_{\epsilon \to 0} \epsilon^2 \ln \bar{\tau}^\epsilon(x_0) = \hat{\mu} \quad , \tag{2.5}$$

where, as before, $\bar{\tau}^\epsilon(x_0) = E_{x_0} \tau^\epsilon(x_0)$ and

$$\begin{aligned} \hat{\mu} &= \min_{y \in \partial \Psi} \frac{1}{2} y^T N y \quad , \\ N &= (DXD^T)^{-1}, \quad AX + XA^T + CC^T = 0 \quad . \end{aligned} \tag{2.6}$$

Constant $\hat{\mu}$ is referred to as the logarithmic residence time of (2.1) in Ψ.

Let $\bar{y}(t, x_0, \hat{x}_0, K, L)$ be the solution of the deterministic system

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} &= \begin{bmatrix} A & BK \\ LE & A + BK - LE \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ \hat{x}(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ \hat{x}_0 \end{bmatrix} \quad , \\ y &= Dx \end{aligned} \tag{2.7}$$

and define

$$\Omega(K, L) = \left\{ \begin{bmatrix} x_0 \\ \hat{x}_0 \end{bmatrix} \in R^{2n} \middle| \bar{y}(t, x_0, \hat{x}_0, K, L) \in \Psi, t \geq 0 \right\} \quad . \tag{2.8}$$

Then, with regard to control system (1.1) and controllers (1.4) or (1.5), Theorem 2.1 allows us to conclude that for sufficiently small ϵ and $\binom{x_0}{\hat{x}_0} \in \Omega(K, L)$, problem (1.2) can be replaced by an alternative problem of selecting the pair (K, L) such that

$$\hat{\mu}(\Psi; K, L) > \mu \tag{2.9}$$

where $\hat{\mu}(\Psi; K, L)$ is the logarithmic residence time of the closed loop system (1.1), (1.4) or (1.1), (1.5) and $\mu = \epsilon^2 \ln T$. This is the problem solved in this paper.

As it was pointed out in the Introduction, the solution of this problem is given in terms of the weak and strong residence time controllability defined precisely below. In order to simplify the notations, we drop argument Ψ in (2.9).

Definition 2.1: (i) System (1.1) is called weakly residence time controllable if for any bounded domain $\Psi \subset \mathbb{R}^p (0 \in \Psi)$ there exists controller (1.4) (or (1.5)) such that $\hat{\mu}(K, L) > 0$;

(ii) System (1.1) is said to be strongly residence time controllable if for any bounded $\Psi \subset \mathbb{R}^p (0 \in \Psi)$ and $\mu > 0$ there exists controller (1.4) (or (1.5)) such that $\hat{\mu}(K, L) > \mu$.

In what follows, we will be assuming that:

A1: (A, C) is disturbable,

A2: (D, A) is detectable,

A3: $FF^T > 0$, and $w(t)$ and $w_1(t)$ are independent Brownian motions,

A4: Transfer matrices $G_s(s) = D(sI - A)^{-1}B, G_n(s) = D(sI - A)^{-1}C$ and $G_{n1}(s) = E(sI - A)^{-1}C$ have full normal rank.

3. NOISELESS MEASUREMENTS CASE

Let $\mathcal{K} \triangleq \{K \in \mathbb{R}^{m\times n} : \ A + BK \text{ is Hurwitz }\}$, $\mathcal{L} \triangleq \{L \in \mathbb{R}^{n\times P} : \ A - LE$ is Hurwitz $\}$ and define the maximal logarithmic residence time of (1.1), (1.4) or (1.1), (1.5) in Ψ as

$$\hat{\mu}^* = \sup_{\substack{K\in\mathcal{K} \\ L\in\mathcal{L}}} \hat{\mu}(K,L) \ . \tag{3.1}$$

Introduce the following hypotheses:

H1: $G_s(s)$ is right invertible and minimum phase.

H2: $G_{n1}(s)$ is left invertible and minimum phase.

H3: There exists an $m \times r$ rational matrix $U(s)$ with no poles in Re $s > 0$ such that $G_n(s) + G_s(s)U(s) = 0$.

H4: There exists a $p \times q$ rational matrix $V(s)$ with no poles in Re $s > 0$ such that $G_n(s) + V(s)G_{n1}(s) = 0$.

Theorem 3.1: System (1.1) is

(i) weakly residence time controllable by controller (1.4) if and only if (A, B) is stabilisable and (E, A) is detectable,

(ii) strongly residence time controllable by controller (1.4) if and only if (A, B) is stabilizable, (E, A) is detectable and either H1 and H4 or H2 and H3 are satisfied.

Proof: See the Appendix.

Remark 3.1: As it was shown in [3], H3 is the condition for strong residence time controllability with respect to the state space feedback $u = Kx$. Furthermore, H1 is a stronger condition than H3. Thus, either H4 or H2 are the additional condition that has to be satisfied when the state space feedback is replaced by the output feedback.

Remark 3.2: In SISO case with $D = E$, Theorem 3.1 implies that for strong residence time controllability $G_s(s)$ should be minimum phase.

A comparison of the fundamental bounds on the residence time achievable by state space (1.3) and output (1.4) feedback can be given as follows:

Consider the closed loop system (1.1), (1.3), i.e.

$$dx = (A + BK)xdt + \epsilon\, Cdw \quad , \tag{3.2}$$

and define as

$$\mu^* = \sup \mu(\Psi; K) \tag{3.3}$$

its maximal logarithmic residence time in Ψ.

Theorem 3.2: Equality $\hat{\mu}^* = \mu^*$ takes place if and only if $G_{n1}(s)$ has a left inverse with no poles in Re $s > 0$.

Proof: See the Appendix.

4. NOISY MEASUREMENTS CASE

Theorem 4.1: Let P be the unique positive definite solution of the (Kalman filter) Riccati equation:

$$AP + PA^T + CC^T - PE^T(FF^T)^{-1}EP = 0 \ . \tag{4.1}$$

Then the maximal logarithmic residence time of the closed loop system (1.1), (1.5) in Ψ satisfies the bound

$$\hat{\mu}^* \leq \min_{y \in \partial \Psi} \frac{1}{2} \, y^T(DPD^T)^{-1}y \ . \tag{4.2}$$

Proof: See the Appendix.

Remark 4.1: It follows, in particular, from Theorem 4.1 that since the upper bound in (4.2) is always finite, system (1.1) with control (1.5) is never strongly residence time controllable. Therefore, the measurement noise in (1.5) has a greater limiting effect on the achievable residence time than the input noise in (1.1).

Theorem 4.2: The upper bound (4.2) is attained if and only if there exists a rational matrix $W(s)$ with no poles in Re $s > 0$ such that

$$G_l(s) + G_s(s)W(s) = 0 \ , \tag{4.3}$$

where $G_s(s)$ is defined as previously and

$$G_l(s) \ = \ D(sI - A)^{-1}\hat{L} \ , \tag{4.4}$$

$$\hat{L} \ = \ PE^T(FF^T)^{-1} \ . \tag{4.5}$$

Proof: See the Appendix.

Remark 4.2: Theorem 4.2 illustrates that the upper bound in (4.2) is attainable. Therefore, it is the best possible upper bound.

5. DESIGN TECHNIQUES

In the two previous sections we have characterized the fundamental bounds on the achievable logarithmic residence time. In this section we develop the controller design techniques that achieve these bounds. First system (1.1) with control (1.5) is considered and then system (1.1) with control (1.4) is addressed. An example is given in Section 6.

To select the pair $\{K, L\}$ that maximizes $\hat{\mu}(K, L)$ assume, for simplicity, that domain Ψ is an ellipsoid

$$\Psi = \{y \in \mathbf{R}^p : \ y^T S y \leq r^2, \ S = S^T > 0\} \ . \tag{5.1}$$

Let $W \in \mathbf{R}^{p \times p}$ be a nonsingular matrix such that $S = W^T W$. Then by direct calculations we obtain:

$$\hat{\mu}(K, L) = \frac{r^2}{2\lambda_{\max}[W D X(K, L) D^T W^T]} \ , \tag{5.2}$$

where $X(K, L)$ is given by

$$\begin{bmatrix} A & BK \\ LE & A + BK - LE \end{bmatrix} \begin{bmatrix} X(K, L) & T(K, L) \\ T^T(K, L) & \widehat{X}(K, L) \end{bmatrix} \tag{5.3}$$

$$+\begin{bmatrix} X(K,L) & T(K,L) \\ T^T(K,L) & \widehat{X}(K,L) \end{bmatrix} \begin{bmatrix} A & BK \\ LE & A+BK-LE \end{bmatrix}^T + \begin{bmatrix} CC^T & 0 \\ 0 & LFF^TL^T \end{bmatrix} = 0$$

Therefore, the pair $\{K, L\}$ is optimal if and only if it minimizes the largest eigenvalue of $\Gamma(K,L) \triangleq WDX(K,L)D^TW^T$. The $\lambda_{max}(\Gamma)$ can be characterized as follows:

Lemma 5.1: Let $\theta \geq 0$ be a scalar, $l \geq 1$ be an integer and select $K_l \in \mathcal{K}$ and $L_l \in \mathcal{L}$ such that

$$\mathrm{Tr}\ \Gamma(K_l, L_l)^l \leq (1+\theta) \inf\{\mathrm{Tr}\ \Gamma(K,L)^l | K \in \mathcal{K}, L \in \mathcal{L}\} \quad . \tag{5.4}$$

Then

$$\lim_{l \to \infty} \lambda_{\max}(\Gamma(K_l, L_l)) = \inf\{\lambda_{\max}(\Gamma(K,L)) | K \in \mathcal{K}, L \in \mathcal{L}\} \tag{5.5}$$

Proof: The proof of this lemma is similar to the proof of Theorem 2.1 in [7]. We omit the details here.

Thus, in order to minimize $\lambda_{\max}(\Gamma)$, we need only to minimize $\mathrm{Tr}\ \Gamma(K,L)^l$, $l = 1, 2, 3 \ldots$. To accomplish this, introduce

$$J_\gamma^l(K,L) = \mathrm{Tr}\ \Gamma(K,L)^l + \gamma\ \mathrm{Tr}\ K\widehat{X}(K,L)K^T \quad , \tag{5.6}$$

where $\widehat{X}(K,L)$ is given by (5.3).

Lemma 5.2: Assume that $K_l^\gamma \in \mathcal{K}$ and $L_l^\gamma \in \mathcal{L}$ minimize $J_\gamma^l(K,L)$. Then

$$\lim_{\gamma \to 0} J_\gamma^l(K_l^\gamma, L_l^\gamma) = \inf_{\substack{K \in \mathcal{K} \\ L \in \mathcal{L}}} \mathrm{Tr}\ \Gamma(K,L)^l \quad . \tag{5.7}$$

Proof: The proof of this lemma is similar to the first part of the proof of the Theorem in [4]. We omit the details here.

From Lemmas 5.1 and 5.2 follows:

Corollary 5.1: Assume that the pair (K_l^γ, L_l^γ) with $K_l^\gamma \in \mathcal{K}$ and $L_l^\gamma \in \mathcal{L}$ minimizes $J_\gamma^l(K, L)$. Then

$$\lim_{l \to \infty} \lim_{\gamma \to 0} \hat{\mu}(K_l^\gamma, L_l^\gamma) = \hat{\mu}^* \quad . \tag{5.8}$$

Thus, K_l^γ and L_l^γ provide the solution to (3.1). A necessary condition for the optimality of (K_l^γ, L_l^γ) in the sense of functional (5.6) can be formulated as follows.

Theorem 5.1: Assume that $K_l^\gamma \in \mathcal{K}$ and $L_l^\gamma \in \mathcal{L}$. Then in order for (K_l^γ, L_l^γ) to minimize $J_\gamma^l(K, L)$ it is necessary that

$$L_l^\gamma = \hat{\mathcal{L}} = PE^T(FF^T)^{-1} \quad , \tag{5.9}$$

$$K_l^\gamma = -\frac{1}{\gamma} B^T Q_l^\gamma \quad , \tag{5.10}$$

where P is given by (4.1) and

$$A^T Q_l^\gamma + Q_l^\gamma A + D^T W^T M_l^\gamma W D - \frac{1}{\gamma} Q_l^\gamma B B^T Q_l^\gamma = 0 \quad , \tag{5.11}$$

$$M_l^\gamma = l(WD(\widehat{X}_l^\gamma + P)D^T W^T)^{l-1} \quad , \tag{5.12}$$

$$(A + BK_l^\gamma)\widehat{X}_l^\gamma + \widehat{X}_l^\gamma(A + BK_l^\gamma)^T + \hat{\mathcal{L}}\hat{\mathcal{L}}^T = 0 \quad . \tag{5.13}$$

Proof: See the Appendix.

Thus, in particular, the optimal observation gain is independent of optimal control while the optimal control gain is a function of optimal observations.

Since (4.1) has a positive definite solution, $L_l^\gamma = \hat{L} \in \mathcal{L}$, $\forall\, \gamma, l$. The following lemma gives a condition for $K_l^\gamma \in \mathcal{K}$.

Lemma 5.3: Assume that $M_l^\gamma > 0$. Then $K_l^\gamma \in \mathcal{K}$.

Proof: See the Appendix.

Remark 5.1: As it follows from Theorem 5.1, the optimal estimator gain $\hat{L}$ given in (5.9) is the Kalman filter gain. Thus, the Kalman filter is optimal in optimization problem (3.1). Moreover, consider the equation for the estimation error $e \triangleq x - \hat{x}$:

$$de = (A - LE)dt + \epsilon\ C\ dw - \epsilon\ LFdw_1 \tag{5.14}$$

and define its logarithmic residence time in any domain $\Lambda \subset \mathbb{R}^n (0 \in \Lambda)$ as $\hat{\mu}(\Lambda; L)$. Then

$$\hat{\mu}(\Lambda; L) = \min_{e \in \partial\Lambda} \frac{1}{2}\, e^T P^{-1}(L) e \quad , \tag{5.15}$$

where $P(L)$ is the positive definite solution of

$$(A - LE)P(L) + P(L)(A - LE)^T + CC^T + LFF^TL^T = 0 \quad . \tag{5.16}$$

Since P given by (4.1) satisfies the inequality

$$P \leq P(L), \quad \forall L \in \mathcal{L} \quad , \tag{5.17}$$

we conclude that

$$\hat{\mu}(\Lambda; \hat{L}) = \min_{e \in \partial\Lambda} \frac{1}{2} e^T P^{-1} e \geq \hat{\mu}(\Lambda; L), \ \forall L \in \mathcal{L} \ . \tag{5.18}$$

Thus, the Kalman filter is optimal in the sense of optimization of the estimation error residence time in every bounded domain of $\boldsymbol{R}^n$.

The optimal control law for system (1.1) with control (1.4) can be obtained from (5.9) - (5.10) by selecting $F = \alpha I$ and letting $\alpha \to 0$. Indeed, since the optimal estimator law for (1.1), (1.5) is the Kalman filter, we know from optimal filtering theory that the optimal (singular) filter for (1.1), (1.4) is obtained in the limit $\alpha \to 0$ (see, e.g., [4]). Therefore, the maximal logarithmic residence time for (1.1), (1.4) is given by

$$\hat{\mu}^* = \lim_{l \to \infty} \lim_{\gamma \to 0} \lim_{\alpha \to 0} \ \hat{\mu}(K_l^{\gamma,\alpha}, L_l^{\gamma,\alpha}) \ , \tag{5.19}$$

where $L_l^{\gamma,\alpha}$ and $K_l^{\gamma,\alpha}$ are given by (5.9) - (5.13) with $FF^T = \alpha^2 I$.

6. EXAMPLE

Consider the second order system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \epsilon \begin{bmatrix} 1 \\ 0 \end{bmatrix} \dot{w} \ , \\ y &= [0 \quad 1] x \ , \\ z &= [1 \quad 0] x + \epsilon \, F \dot{w}_1 \ . \end{aligned} \tag{6.1}$$

For this system

$$G_s(s) = \frac{s}{s^2+1} \ , \ G_n(s) = \frac{-1}{s^2+1} \ , \ G_{n1}(s) = \frac{s}{s^2+1} \ . \tag{6.2}$$

Thus, since $G_s(s) = G_{n1}(s)$ is minimum phase, this system is *srt*-controllable by controller (1.4) when $F \equiv 0$.

Assume that $F \neq 0$. Then, by Theorem 4.1, the logarithmic residence time in the interval $\Psi = (-a, b)$, $a, b > 0$, is bounded by

$$\min_{y \in \partial\Psi} \frac{1}{2} \, y^T (DPD^T)^{-1} \, y = \frac{(\min(a,b))^2}{2|F|} \ . \tag{6.3}$$

Furthermore, when $a = b$, the (sub)optimal controller can be calculated using (5.9) - (5.13) to be

$$\hat{L} = \begin{bmatrix} \frac{1}{|F|} \\ 0 \end{bmatrix} \ , \ K_l^\gamma = -[0 \quad K_2] \tag{6.4}$$

where $K_2 > 0$ satisfies the equation

$$\frac{K_2^2 \gamma}{l|F|^{l-1}} = \left(1 + \frac{|F|}{2K_2}\right)^{l-1} \ . \tag{6.5}$$

The logarithmic residence time with this control is

$$\hat{\mu}(K_l^\gamma, \hat{L}) = \frac{a^2}{2|F|} \cdot \frac{2K_2}{2K_2 + |F|} \ . \tag{6.6}$$

Note that $\hat{\mu}(K_l^\gamma, \hat{L})$ is the upper bound in (6.3) multiplied by the factor

$$\rho = \frac{2K_2}{2K_2 + |F|} \ . \tag{6.7}$$

Thus, in order to obtain logarithmic residence time as close as desired to the maximal value, (6.3), (6.7) can be used to calculate the necessary K_2 (for a given ρ) and l and γ can be determined from (6.5).

As $\gamma \to 0$ equation (6.5) simplifies considerably. Indeed, in this case $K_2 \to \infty$ and, thus, for small γ (6.5) becomes

$$\frac{K_2^2\gamma}{l|F|^{l-1}} \simeq 1 \ . \tag{6.8}$$

Therefore,

$$K_2 \simeq \sqrt{\frac{l}{\gamma}}\,|F|^{\frac{l-1}{2}} \ . \tag{6.9}$$

7. CONCLUSIONS

It is shown in this paper that the observer based output feedback can be efficiently used for pointing of linear systems subject to both input and measurement noise. The fundamental bounds on the achievable precision of pointing depend on the locations of the right half plane zeros of the various transfer functions involved. Roughly speaking, the best precision of pointing is obtained for minimum phase systems. Any desired precision of aiming is attainable only if no measurement noise is present. Therefore, the effect of the measurement noise on the achievable precision of aiming is more detrimental than that of the input noise.

APPENDIX

Proof of Theorem 3.1: The proof of (i) parallels the proof of Theorem 3.1 in [3]. We omit the details here. In order to prove (ii) we first derive the inequality

$$\frac{r^2}{2 \text{ Tr } DX(K,L)D^T} \le \hat{\mu}(K,L) \le \frac{pR^2}{2 \text{ Tr } DX(K,L)D^T} \; , \qquad \text{(A.1)}$$

where $K \in \mathcal{K}$, $L \in \mathcal{L}$. To get the left inequality note that

$$\begin{aligned}
\hat{\mu}(K,L) &\ge \frac{1}{2}\, \lambda_{\min}[(DX(K,L)D^T)^{-1}] \min_{y \in \partial \Psi} y^T y \\
&= \frac{r^2}{2\lambda_{\max}[DX(K,L)D^T]} \\
&\ge \frac{r^2}{2 \text{ Tr } DX(K,L)D^T} \; .
\end{aligned}$$

For the right inequality we have ($R^2 = \max_{y \in \partial \Psi} y^T y$, $B(0,R) = \{y | y^T y \le R^2\}$)

$$\begin{aligned}
\hat{\mu}(K,L) &= \min_{y \in \partial \Psi} \frac{1}{2}\, y^T (DX(K,L)D^T)^{-1} y \\
&\le \min_{y \in \partial B(0,R)} \frac{1}{2}\, y^T (DX(K,L)D^T)^{-1} y \\
&= \frac{1}{2}\, \lambda_{\min}[(DX(K,L)D^T)^{-1}] R^2 \\
&= \frac{R^2}{2\lambda_{\max}[DX(K,L)D^T]} \\
&\le \frac{pR^2}{2 \text{ Tr } DX(K,L)D^T} \; .
\end{aligned}$$

It follows from (A.1) that $\hat{\mu}^* = \infty$ is equivalent to

$$\inf_{\substack{K\in\mathcal{K}\\ L\in\mathcal{L}}} \text{ Tr } DX(K,L)D^T = 0 \ . \tag{A.2}$$

Next note that it follows from linear quadratic theory [4], [5] that

$$\inf_{\substack{K\in\mathcal{K}\\ L\in\mathcal{L}}} \text{ Tr } DX(K,L)D^T = \lim_{\substack{\gamma\to 0\\ \alpha\to 0}} \text{ Tr } DX(K^\gamma, L^\gamma)D^T \tag{A.3}$$

where

$$K^\gamma = -\frac{1}{\gamma} B^T Q^\gamma, \quad A^T Q^\gamma + Q^\gamma A + D^T D - \frac{1}{\gamma} Q^\gamma B B^T Q^\gamma = 0 \ , \tag{A.4}$$

$$L^\alpha = -\frac{1}{\alpha} P^\alpha E^T, \quad A P^\alpha + P^\alpha A + C C^T - \frac{1}{\alpha} P^\alpha E^T E P^\alpha = 0 \ . \tag{A.5}$$

Furthermore,

$$\begin{aligned}\lim_{\substack{\gamma\to 0\\ \alpha\to 0}} \text{ Tr } DX(K^\gamma, L^\gamma)D^T &= \lim_{\substack{\gamma\to 0\\ \alpha\to 0}} \text{ Tr } (DP^\alpha D^T + \alpha L^{\alpha^T} Q^\gamma L^\alpha) \\ &= \lim_{\substack{\gamma\to 0\\ \alpha\to 0}} \text{ Tr } (C^T Q^\gamma C + \gamma\, K^\gamma P^\alpha K^{\gamma^T}) \ . \end{aligned} \tag{A.6}$$

Therefore, with $\tilde{C} = \lim_{\alpha\to 0} \sqrt{\alpha}\ L^\alpha$ and $\hat{D} = \lim_{\alpha\to 0} \sqrt{\gamma}\ K^\gamma$, we have

$$\begin{aligned}\inf_{\substack{K\in\mathcal{K}\\ L\in\mathcal{L}}} \text{ Tr } DX(K,L)D^T &= \text{ Tr } (DP^0 D^T + \tilde{C}^T Q^0 \tilde{C}) \\ &= \text{ Tr } (\hat{D} P^0 \hat{D}^T + C^T Q^0 C) \ . \end{aligned} \tag{A.7}$$

Each of the terms Tr DP^0D^T, Tr $\tilde{C}^TQ^0\tilde{C}$, Tr $\hat{D}P^0\hat{D}^T$ and Tr C^TQ^0C is nonnegative. Thus system (1.1) with control (1.4) is strongly residence time controllable if and only if all four terms are zero.

It was shown in [3] that Tr $C^T Q^0 C = 0$ if and only if there exists a rational matrix $U(s)$, with no poles in Re $s > 0$, such that

$$G_n(s) + G_s(s)U(s) = 0 \ . \tag{A.8}$$

Similarlily, Tr $\tilde{C}^T Q^0 \tilde{C} = 0$, Tr $DP^0D^T = 0$ and Tr $\widehat{D}P^0\widehat{D}^T = 0$ if and only if there exist rational matrices $\tilde{U}(s)$, $V(s)$ and $\hat{V}(s)$, with no poles in Re $s > 0$, such that

$$\begin{aligned} \tilde{G}_n(s) + G_s(s)\tilde{U}(s) &= 0 \ , &\text{(A.9)}\\ G_n(s) + V(s)G_{n1}(s) &= 0 \ , &\text{(A.10)}\\ \hat{G}_n(s) + \hat{V}(s)G_{n1}(s) &= 0 &\text{(A.11)}\end{aligned}$$

where

$$\begin{aligned} \tilde{G}_n(s) &= D(sI - A)^{-1}\,\tilde{C} \ , &\text{(A.12)}\\ \hat{G}_n(s) &= \widehat{D}(sI - A)^{-1}\,C \ . &\text{(A.13)}\end{aligned}$$

Now, if H1 is satisfied then $U(s) = -G_s^{-1}(s)G_n(s)$ and $\tilde{U}(s) = -G_s^{-1}(s)\tilde{G}_n(s)$ ($G_s^{-1}(s)$ is the right inverse of $G_s(s)$) are both without poles in Re $s > 0$ and satisfy (A.8) and (A.9). Therefore, Tr $C^T Q^0 C =$ Tr $\tilde{C}^T Q^0 \tilde{C} = 0$. Furthermore, in this case $D^T D = \widehat{D}^T \widehat{D}$ (see, e.g., [4]) and, thus, H4 implies that $0 =$ Tr $DP^0D^T =$ Tr $P^0 D^T D =$ Tr $P^0 \widehat{D}^T \widehat{D} =$ Tr $\widehat{D} P^0 \widehat{D}^T$. Therefore, by (A.7) the system is

strongly residence time controllable. Similarily, if H3 is satisfied, then $V(s) = -G_n(s)G_{n1}^{-1}(s)$ and $\hat{V}(s) = -G_n(s)G_{n1}^{-1}(s)$ are both without poles in Re $s > 0$ and, thus, Tr $DP^0D^T =$ Tr $\widehat{D}P^0\widehat{D}^T = 0$. Furthermore, $CC^T = \tilde{C}\tilde{C}^T$ and, therefore, H3 implies that $0 =$ Tr $C^TQ^0C =$ Tr $\tilde{C}^TQ^0\tilde{C}$. This proves the sufficiency part of the theorem.

Assume now that (1.1), (1.4) is strongly residence time controllable. Then (A.8) - (A.11) are satisfied and, thus, H3 and H4 are true. Note that the existence of $U(s)$ such that (A.8) is satisfied and A4 imply that $m \geq \min(p,r)$. Similarly, the existence of $V(s)$ and A4 imply that $q \geq \min(p,r)$. Assume $p \leq r$. Then $m \geq p$ and, thus, $G_s(s)$ is right invertible. Similarly, if $p \geq r$, then $q \geq r$ and $G_{n1}(s)$ is left invertible. Next, it can be shown that (A.10) implies that $G_n(s)G_n^T(-s) = \tilde{G}_n(s)\tilde{G}_n^T(-s)$. Furthermore, $\tilde{G}_n(s)$ has no zeroes in Re $s > 0$ (see, e.g. [6]). Similarly, (A.8) implies that $G_n^T(-s)G_n(s) = \hat{G}_n^T(-s)\hat{G}_n(s)$ and $\hat{G}_n(s)$ has no zeroes in Re $s > 0$. Thus, if $p \leq r$, i.e. $G_s(s)$ is right invertible, then it follows from (A.10) and (A.9) that $G_s(s)$ has no zeroes in Re $s > 0$. Thus H1 is satisfied. Similarily, if $p \geq r$, then (A.8) and (A.11) imply that $G_{n1}(s)$ is left invertible and minimum phase, i.e., H2 is true. **Q.E.D.**

Proof of Theorem 3.2: Let $\mu(K)$ be the logarithmic residence time of (3.12). Then, obviously, for any $K \in \mathcal{K}$ and $L \in \mathcal{L}$ we have

$$\hat{\mu}(K,L) \leq \mu(K) \tag{A.14}$$

and, thus,

$$\sup_{L\in\mathcal{L}} \hat{\mu}(K,L) \leq \mu(K) \quad . \tag{A.15}$$

Furthermore, using a similar argument to the one in the proof of Theorem 4.1 (see below) we have

$$\sup_{L\in\mathcal{L}} \hat{\mu}(K,L) = \lim_{\alpha\to 0} \hat{\mu}(K,L^\alpha) \tag{A.16}$$

$$L^\alpha = P^\alpha E^T \ , \ AP^\alpha + P^\alpha A^T + CC^T - \frac{1}{\alpha}\, P^\alpha E^T E P^\alpha = 0 \quad . \tag{A.17}$$

Thus, we want to show that left invertibility and minimum phase of $G_{n1}(s)$ is necessary and sufficient for

$$\begin{aligned} \lim_{\alpha\to 0} \hat{\mu}(K,L^\alpha) &= \min_{y\in\partial\Phi} \frac{1}{2}\, y^T (D(\hat{X}(K) + P^0)D^T)^{-1} y \\ &= \mu(K) \end{aligned} \tag{A.18}$$

where

$$(A+BK)\hat{X}(K) + \hat{X}(K)(A+BK) + \tilde{C}\tilde{C}^T = 0 \tag{A.19}$$

for all $K \in \mathcal{K}$. However, since

$$\mu(K) = \min \ \frac{1}{2}\, y^T (DX(K)D^T)^{-1} y \quad , \tag{A.20}$$

$$(A+BK)X(K) + X(K)(A+BK)^T + CC^T = 0 \quad , \tag{A.21}$$

it follows that (A.18) is true if and only if $DP^0D^T = 0$ and $CC^T = \tilde{C}\tilde{C}^T$. These are exactly the necessary and sufficient conditions for $G_{n1}(s)$ to be left invertible and minimum phase. **Q.E.D.**

Proof of Theorem 4.1: It is straight forward to show that $X(K,L) \geq P$ (see, e.g. equation (A.25) below). Therefore, since

$$\hat{\mu}(K,L) = \min_{y\in\partial\Phi} \frac{1}{2}\, y^T(DX(K,L)D^T)^{-1}y \quad , \tag{A.22}$$

inequality (4.2) follows. **Q.E.D.**

Proof of Theorem 4.2: The logarithmic residence time in a system with the optimal estimator gain $\hat{L} = PE^T(FF^T)^{-1}$ is

$$\hat{\mu}(K,\hat{L}) = \min_{y\in\partial\Phi} \frac{1}{2}\, y^T(D(\widehat{X}(K) + P)D^T)^{-1}y \tag{A.23}$$

where

$$(A+BK)\widehat{X}(K) + \widehat{X}(K) + \hat{L}\hat{L}^T = 0 \quad . \tag{A.24}$$

Thus, the upper bound (4.2) is attained if and only if $\inf_{K\in K}$ Tr $D\widehat{X}(K)D^T = 0$. However, by the same argument as was used in the proof of Theorem 3.1, this happens if and only if (4.3) is satisfied. **Q.E.D.**

Proof of Theorem 5.1: Let $\hat{L}$ be the Kalman filter gain (5.9) and define

$$d\tilde{x} = (A\tilde{x} + BK\hat{x})dt + \hat{L}(dz - E\tilde{x}dt) \tag{A.25}$$

where $\hat{x}$ is the estimate (1.5) for an arbitrary L. Then [5]

$$X(K,L) = \widetilde{X}(K,L) + P \tag{A.26}$$

where P satisfies (4.1) and $\widetilde{X}$ is given by

$$\begin{pmatrix} A & BK \\ LE & A+BK-LE \end{pmatrix} \begin{pmatrix} \widetilde{X} & Z \\ Z^T & \widehat{X} \end{pmatrix} + \begin{pmatrix} \widetilde{X} & Z \\ Z^T & \widehat{X} \end{pmatrix} \begin{pmatrix} A & BK \\ LE & A+BK-LE \end{pmatrix}^T$$

$$+ \begin{pmatrix} \hat{L}FF^T\hat{L}^T & \hat{L}FF^TL^T \\ LFF^T\hat{L}^T & LFF^TL^T \end{pmatrix} = 0 \ . \tag{A.27}$$

Define

$$\begin{pmatrix} \widetilde{X} & X_1 \\ X_1^T & X_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix} \begin{pmatrix} \widetilde{X} & Z \\ Z^T & \widehat{X} \end{pmatrix} \begin{pmatrix} I & I \\ 0 & -I \end{pmatrix} \ . \tag{A.28}$$

Then

$$\begin{pmatrix} A+BK & -BK \\ 0 & A-LE \end{pmatrix} \begin{pmatrix} \widetilde{X} & X_1 \\ X_1 & X_2 \end{pmatrix} + \begin{pmatrix} \widetilde{X} & X_1 \\ X_1^T & X_2 \end{pmatrix} \begin{pmatrix} A+BK & -BK \\ 0 & A-LE \end{pmatrix}^T$$

$$+ \begin{pmatrix} \hat{L}FF^T\hat{L}^T & \hat{L}FF^T(\hat{L}-L)^T \\ (\hat{L}-L)FF^T\hat{L} & (\hat{L}-L)FF^T(\hat{L}-L)^T \end{pmatrix} = 0 \ . \tag{A.29}$$

In order to show that $K_l^\gamma, \hat{L}$ satisfy the necessary conditions for minimizing $J_\gamma^l(K,L)$ we have to show that $\left(F = \binom{K}{L^T}\right)$

$$\frac{\partial J_\gamma^l(F)}{\partial F} = 0 \tag{A.30}$$

gives $F = F_l^\gamma = \binom{K_l^\gamma}{\hat{L}^T}$. (A.30) is equivalent to showing that

$$\frac{d}{d\epsilon}\, J_\gamma^l(F+\epsilon\Delta F)\bigg|_{\epsilon=0} = \left\langle \frac{\partial J_\gamma^l}{\partial F}\,(F),\ \Delta F \right\rangle = \mathrm{Tr}\ \Delta F^T \frac{\partial J_\gamma^l}{\partial F}(F) = 0 \tag{A.31}$$

for all $\Delta F = \binom{\Delta K}{\Delta L^T}$. In order to simplify notation we assume $WD = I$. Evaluating $\frac{d}{d\epsilon} J_\gamma^l(F + \epsilon \Delta F)\Big|_{\epsilon=0}$ gives

$$\begin{aligned}\frac{d}{d\epsilon}\, J_\gamma^l(F + \epsilon\Delta F)\Big|_{\epsilon=0} &= l \text{ Tr } X^{l-1}X' + \gamma \text{ Tr } K\widehat{X}'K^T \\ &\quad + \gamma \text{ Tr } \Delta K \widehat{X} K^T + \gamma \text{ Tr } K\widehat{X}\Delta K^T \quad . \qquad \text{(A.32)}\end{aligned}$$

where

$$X' = \frac{d}{d\epsilon}\, X(K + \epsilon\Delta K,\ L + \epsilon \Delta L)\Big|_{\epsilon=0} \quad , \qquad \text{(A.33)}$$

$$\widehat{X}' = \frac{d}{d\epsilon}\, \widehat{X}(K + \epsilon\Delta K, L + \epsilon \Delta L)\Big|_{\epsilon=0} \quad . \qquad \text{(A.34)}$$

From (A.26) and (A.28) we get $\widehat{X} = \widetilde{X} - X_1^T - X_1 + X_2$ and

$$\widehat{X}' = \widetilde{X}' - X_1'^T - X_1' + X_2' \quad , \qquad \text{(A.35)}$$

$$X' = \widetilde{X}' \quad \text{(since } P = \text{const.)} \qquad \text{(A.36)}$$

where

$$\widetilde{X}' = \frac{d}{d\epsilon}\, \widetilde{X}\,\Big|_{\epsilon=0} \quad , \qquad \text{(A.37)}$$

$$X_1' = \frac{d}{d\epsilon}\, X_1\Big|_{\epsilon=0} \quad , \qquad \text{(A.38)}$$

$$X_2' = \frac{d}{d\epsilon}\, X_2\Big|_{\epsilon=0} \quad . \qquad \text{(A.39)}$$

Using this in (A.32) gives

$$\frac{d}{d\epsilon}\, J_\gamma^l(F + \epsilon\Delta F)\Big|_{\epsilon=0} = l \text{ Tr } X^{l-1}\widetilde{X}'$$

$$+ \quad \gamma \ \mathrm{Tr}\ K^T K(\widetilde{X}' - X_1'^T - X_1' + X_2') \tag{A.40}$$

$$+ \quad \gamma \ \mathrm{Tr}\ \widehat{X} K^T \Delta K + \gamma \ \mathrm{Tr}\ \Delta K^T K \widehat{X} \ .$$

From (A.29) we get the following equations for $\widetilde{X}, X_1$ and X_2

$$(A + BK)\widetilde{X} + \widetilde{X}(A + BK)^T - BKX_1 - X_1^T K^T B^T + \hat{L}FF^T\hat{L}^T = 0 \ , \tag{A.41}$$

$$(A + BK)X_1 + X_1(A - LE)^T - BKX_2 + \hat{L}FF^T(\hat{L} - L)^T = 0 \ , \tag{A.42}$$

$$(A - LE)X_2 + X_2(A - LE)^T + (\hat{L} - L)FF^T(\hat{L} - L)^T = 0 \ . \tag{A.43}$$

Thus, $\widetilde{X}', X_1'$ and X_2' satisfy

$$(A + BK)\widetilde{X}' + \widetilde{X}'(A + BK)^T + B\Delta K\widetilde{X} + \widetilde{X}\Delta K^T B^T$$

$$- BKX_1' - X_1'^T K^T B^T - B\Delta KX_1 - X_1^T \Delta K^T B^T = 0 \ , \tag{A.44}$$

$$(A + BK)X_1' + X_1'(A - LE)^T + B\Delta KX_1 - X_1E^T\Delta L^T$$

$$- BKX_2' - B\Delta KX_2 - \hat{L}FF^T\Delta L^T = 0 \ , \tag{A.45}$$

$$(A - LE)X_2' + X_2'(A - LE)^T - \Delta LEX_2 - X_2E^T\Delta L^T$$

$$- (\hat{L} - L)^T FF^T \Delta L^T - \Delta LFF^T(\hat{L} - L)^T = 0 \ . \tag{A.46}$$

Next we rewrite (A.40) using (A.35) and the adjoint equation for (A.44). This gives

$$\frac{d}{d\epsilon} J_\gamma^l(F + \epsilon\Delta F)\bigg|_{\epsilon=0} = \mathrm{Tr}\ (\widetilde{X}QB + \gamma\widehat{X}K^T - X_1QB)\Delta K$$

$$+\mathrm{Tr}\ \Delta K^T(B^TQ\widetilde{X}+\gamma K\widehat{X}-B^TQX_1^T)-\mathrm{Tr}(QB+\gamma K^T)KX_1'$$

$$-\ \mathrm{Tr}\ X_1'^T K^T(B^TQ+\gamma K)+\gamma\ \mathrm{Tr}\ K^TKX_2' \qquad \text{(A.47)}$$

where

$$(A+BK)^TQ+Q(A+BK)+lX^{l-1}+\gamma K^TK=0\ . \qquad \text{(A.48)}$$

Now, it follows from (A.46) and the last term in (A.47) that in order for (A.47) to be zero for any ΔL it is necessary that $X_2'=0$. Thus, $EX_2+FF^T(\hat{L}-L)^T=0$. Substituting $\hat{L}-L=-X_2E^T(FF^T)^{-1}$ into (A.43) gives $X_2=0$. Therefore $L=\hat{L}$. Furthermore, with $L=\hat{L}$ and $X_2=0$ it follows from (A.42) that for any $K\in\mathcal{K}$ we have $X_1=0$. Therefore $\widetilde{X}=\widehat{X}$ and the first two terms on the right hand side of (A.47) give $\gamma K+B^TQ=0$. However, this makes the third and fourth terms in the right hand side of (A.47) also equal to zero. Therefore, in order for (A.47) to be identically zero for any ΔF we must have $L=L$ and $K=-\frac{1}{\gamma}B^TQ$.

Finally, substituting $K=K_l^\gamma=-\frac{1}{\gamma}B^TQ$ into (A.48) gives (5.11) and (A.26), (A.41) with $\widetilde{X}=\widehat{X}$ gives (5.13). **Q.E.D.**

Proof of Lemma 5.3: Note that if $M_l^\gamma>0$ then $M_l^\gamma=N_l^{\gamma T}N_l^\gamma$ for some non-singular N_l^γ. Furthermore, since (D,A) is detectable it follows that $(N_l^\gamma WD,A)$ is detectable. Thus, $Q_l^\gamma\geq 0$ and $K_l^\gamma\in\mathcal{K}$. **Q.E.D.**

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SECTION III

AIMING CONTROL: RESIDENCE PROBABILITY AND (D, T)-STABILITY

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Abstract

In this paper, the problem of aiming control is formulated and analyzed in terms of the residence probability measure. Specifically, the notion of residence probability in a domain is introduced and its asymptotic expression is derived for linear systems with small, additive white noise. The associated notion of (D,T)-stability, which characterizes the performance of stochastic systems with no equilibrium points, is introduced and investigated. Finally, the controllability of residence probability is studied and the necessary and sufficient conditions for (D,T)-stabilizability are derived. The development is based on the asymptotic large deviations theory.

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1. INTRODUCTION AND STATEMENT OF THE PROBLEM

Consider a system described by the Ito stochastic differential equation:

$$dx = (Ax + Bu)dt + \epsilon\, Cdw \ , \quad x(0) = x_0 \ , \tag{1.1}$$

where $x \in R^n, u \in R^m, 0 < \epsilon \ll 1$ and w is a standard r-dimensional Brownian motion. Let $D \subset R^n$ be an open bounded domain with 0 in its interior and $T < \infty$ a positive number. Consider the following problem:

Given system (1.1) and the pair (D, T), find a feedback law

$$u = Kx \tag{1.2}$$

and an open set $D_0 \subset D$ such that the closed loop system (1.1), (1.2) has the following property:

$$x(t, x_0) \in D, \ \forall\, t \in [0, T], \ \forall\, x_0 \in [D_0] \ ,$$

where $[D_0]$ is the closure of D_0.

This problem is referred to as the problem of *aiming control*. Such a problem arises in a number of applications where the goal is to accomplish a certain task during a specified period (T) with a specified accuracy (D). Examples include the telescope pointing problem [1], robot arm and laser beam pointing [2], [3], missile terminal guidance [4], etc.

It is well known, however, that

$$\mathrm{Prob}\{x(t,x_0) \in \partial D \text{ for some } t > 0 | x_0 \in [D_0]\} = 1 \quad ,$$

where ∂D is the boundary of D and D_0 is any open subset of D, [5]-[7]. Therefore, the aiming process specifications, (D,T), cannot be met exactly and some probabilistic meaning should be attached to their interpretation. This can be accomplished using the notion of the *first passage time*:

$$\tau_{x_0} = \inf\{t > 0 : \ x(t) \in \partial D | x_0 \in [D_0]\} \quad . \tag{1.3}$$

Specifically, the problem of aiming control can be re-formulated in the following two probabilistic settings:

Residence time control: Given (1.1) and a pair (D,T), find a feedback law (1.2) and an open set $D_0 \subset D$ such that

$$E[\tau_{x_0}] \geq T \ , \quad \forall \ x_0 \in [D_0] \quad . \tag{1.4}$$

Residence probability control: Given (1.1), a pair (D,T) and a constant $0 < p < 1$, find a feedback law (1.2) and an open set $D_0 \subset D$ such that

$$\mathrm{Prob}\{\tau_{x_0} > T\} > p \ , \quad \forall \ x_0 \in [D_0] \quad . \tag{1.5}$$

The residence time control problem (1.4) and its generalizations has been analyzed in [8]-[12]. In these publications, the fundamental bounds on the achievable values of $E[\tau_{x_0}]$ have been investigated and the methods for controllers design, compatible with these bounds, have been developed.

The residence probability control (1.5) appears to be a stronger reformulation of the aiming control problem than the residence time control. Indeed, since τ_{x_0} is non-negative random variable, the Markov inequality gives:

$$\text{Prob}\{\tau_{x_0} \geq T\} \leq \frac{E[\tau_{x_0}]}{T} \quad .$$

Therefore, if $\text{Prob}\{\tau_{x_0} \geq T\} \geq p$, the estimate for $E[\tau_{x_0}]$ follows immediately:

$$E[\tau_{x_0}] \geq pT \quad .$$

On the other hand, it is possible to show (see Appendix 4) that for any D, D_0, T and $0 < p < 1$ there exists a feedback law (1.2) and $x_0 \in D_0$ such that the closed loop system (1.1), (1.2) has the following property:

$$E[\tau_{x_0}] > T \quad ,$$

$$\text{Prob}\{\tau_{x_0} > T\} < p \quad .$$

This implies that the closed loop system may exhibit a performance as good as desired from the residence time point of view and as bad as desired from the point of view of the residence probability.

These observations justify the problem of residence probability control and, in addition, indicate that it has a more complicated mathematical structure than residence time control problem.

This paper is devoted to the investigation of the controllability properties of residence probability, i.e., to the question on when there exists a feedback

law $u = Kx$ that solves the residence probability control problem. The related question of design of the residence probability controllers will be addressed in a companion paper. As it was the case of [8]-[12], the development is based on the large deviations theory of [7].

The idea of utilizing the residence probability as a measure of control systems performance is not new. Apparently, it was first introduced in [13] and then analyzed in [14]-[17]. The approach of [14] is based on the stochastic Liapunov functions and as a result the estimates obtained are quite conservative. Indeed, if $(A + BK)$ and C of (1.1), (1.2) have the form of Example 4, Chapter 3 of [14], i.e.,

$$A + BK = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \ , \ C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and D is given by

$$D = \{x \in R^2 : \ \frac{3}{2}\, x_1^2 + x_1 x_2 + x_2^2 \leq 2\} \ ,$$

then choosing, for instance, $x_0 = [1\ 0]^T$, the approach of [14] results in

$$\text{Prob}\{\tau_{x_0} \leq T\} \leq 1 - 0.25e^{-\frac{\epsilon^2 T}{2}} \ . \tag{1.6}$$

Using the asymptotic method of this paper, for the same example we obtain

$$\text{Prob}\{\tau_{x_0} \leq T\} \leq e^{-\frac{0.2818}{\epsilon^2}} \ , \ \ 0 < T < \infty \ . \tag{1.7}$$

Obviously, (1.6) and (1.7) have different asymptotic behavior: when ϵ is very

small, (1.7) gives

$$\text{Prob}\{\tau_{x_0} \leq T\} \simeq 0 \ ,$$

whereas (1.6) results in

$$\text{Prob}\{\tau_{x_0} \leq T\} \leq 0.75 \ .$$

Thus, the asymptotic approach, although restricted by the condition $\epsilon \ll 1$, is advantageous in comparison with the Liapunov functions method.

Note that along with [6] and [7], there are other asymptotic techniques for calculating the residence probability (see, for instance, [18]-[20]). However, the problem of controllability of residence probability has not been explored. It is done in this paper.

The structure of the paper is as follows: In Section 2 an asymptotic formula for residence probability in a domain is derived. In Section 3, the notion of (D,T)-stability, that characterizes the behavior of stochastic systems with no equilibrium points, is introduced and analyzed. Section 4 presentes the conditions for residence probability controllability and (D,T)-stabilizability. In Section 5, the conclusions are formulated. The proofs are given in Appendicies 1-4.

2. RESIDENCE PROBABILITY IN A DOMAIN

Consider the Ito system

$$dx = Axdt + \epsilon\, Cdw \ , \qquad x(0) = x_0 \ , \tag{2.1}$$

where, as before, $x \in R^n$, $0 < \epsilon \ll 1$ and w is a standard r-dimensional Brownian motion. Let $D \subset R^n$ and $D_0 \subset D$ be open bounded sets with 0 in their interior and smooth boundaries ∂D and ∂D_0, respectively. The first passage time of the trajectory originating at x_0 is

$$\tau_{x_0} = \inf\{t \geq 0 : \ x(t) \in \partial D | x_0 \in [D_0]\} \ . \tag{2.2}$$

As a random variable, τ_{x_0} is characterized by its probability distribution, i.e., $\text{Prob}\{\tau_{x_0} \leq T\}$. Based on this distribution, the first passage probability of the trajectories originating in $[D_0]$ can be defined as follows:

$$P_{D_0}\{\tau \leq T\} \triangleq \max_{x_0 \in [D_0]} \text{Prob}\{\tau_{x_0} \leq T\} \ . \tag{2.3}$$

Then the *residence probability in the domain* is

$$\begin{aligned} P_{D_0}\{\tau > T\} &\triangleq 1 - P_{D_0}\{\tau \leq T\} \\ &= \min_{x_0 \in [D_0]} \text{Prob}\{\tau_{x_0} > T\} \ . \end{aligned} \tag{2.4}$$

These probabilities play a crucial role in the development that follows. They are characterized next:

Theorem 2.1: Assume that (A, C) is disturbable. Then

$$\lim_{\epsilon \to 0} \epsilon^2 \ln P_{D_0}\{\tau \leq T\} = - \min_{x_0 \in [D_0]} \min_{\substack{y \in \partial D \\ 0 \leq t \leq T}} \frac{1}{2}(y - e^{At}x_0)^T X^{-1}(t)(y - e^{At}x_0) \ , \tag{2.5}$$

where

$$\dot{X}(t) = AX(t) + X(t)A^T + CC^T \ , \ X(0) = 0 \ . \tag{2.6}$$

Proof: See Appendix 1.

The interpretation of this result is as follows: Let $-\varphi(D_0)$ denote the right hand side of (2.5), i.e.,

$$\varphi(D_0) \triangleq \min_{x_0 \in [D_0]} \min_{\substack{y \in \partial D \\ 0 \leq t \leq T}} \frac{1}{2}(y - e^{At}x_0)^T X^{-1}(t)(y - e^{At}x_0) \quad . \tag{2.7}$$

Then, according to Theorem 2.1, if ϵ is sufficiently small, for any $\delta > 0$,

$$e^{-\frac{\varphi(D_0)+\delta}{\epsilon^2}} \leq P_{D_0}\{\tau \leq T\} \leq e^{-\frac{\varphi(D_0)-\delta}{\epsilon^2}} \quad ,$$

i.e. $P_{D_0}\{\tau \leq T\}$ is *logarithmically equivalent* to $e^{-\frac{\varphi(D_0)}{\epsilon^2}}$. Due to this reason, $\varphi(D_0)$ is referred to as the *logarithmic first passage probability*. Obviously, the residence probability in the domain, i.e., $P_{D_0}\{\tau > T\}$, is logarithmically equivalent to $1 - e^{-\frac{\varphi(D_0)}{\epsilon^2}}$.

As it follows from the above, the logarithmic first passage probability is independent of the distribution of initial points x_0 in D_0. In addition, $\varphi(D_0)$ is a coordinate-free characterization of system's performance. Indeed, consider a similarity transformation

$$x = Q\hat{x}, \quad \det Q \neq 0 \tag{2.7}$$

and the system

$$\begin{aligned} d\hat{x} &= \hat{A}\hat{x}dt + \epsilon\, \hat{C}dw \ , \\ \hat{A} &= Q^{-1}AQ \ , \ \ \hat{C} = Q^{-1}C \end{aligned} \tag{2.8}$$

Let

$$\widehat{D} = \{\hat{x} \in R^n : Q\hat{x} \in D\} ,$$
$$\widehat{D}_0 = \{\hat{x} \in R^n : Q\hat{x}_0 \in D_0\} .$$

and $\hat{\varphi}(\widehat{D}_0)$ be the logarithmic first passage probability from $\widehat{D}$, i.e.,

$$\hat{\varphi}(\widehat{D}_0) = \min_{\hat{x} \in [\widehat{D}_0]} \min_{\substack{\hat{y} \in \partial \widehat{D} \\ 0 \leq t \leq T}} \frac{1}{2}(\hat{y} - e^{\hat{A}t}\hat{x}_0)^T \widehat{X}^{-1}(t)(\hat{y} - e^{\hat{A}t}\hat{x}_0) .$$

Then

Lemma 2.1: $\hat{\varphi}(\widehat{D}_0) = \varphi(D_0)$.

Proof: See Appendix 1.

This property will be used in Section 4 to establish the upper bound of the achievable logarithmic first passage probability for system (1.1).

3. (D,T)-STABILITY

If a stochastic system has an equilibrium point, its stability can be characterized by the usual notion of Liapunov stability modified in an appropriate stochastic sense [14], [21]. If the system does not have equilibria, as is the case for (2.1), the Liapunov stability does not apply. In this situation, the notion of first passage time could be used to describe its "stability" features. One way to accomplish this is as follows:

Definition 3.1: System (2.1) is said to be (D,T)-stable with probability $0 < p < 1$ if there exists an open set $D_0 \subset D$ such that

$$P_{D_0}\{\tau > T\} > p \tag{3.1}$$

or, equivalently,

$$P_{D_0}\{\tau \leq T\} \leq 1 - p \quad , \tag{3.2}$$

where $P_{D_0}\{\tau > T\}$ and $P_{D_0}\{\tau \leq T\}$ are defined in (2.4) and (2.3), respectively.

In this Definition, set D may be interpreted as a safe operating region, T as a desired operating time, and D_0 as an initial, "lock in", set.

Applying this definition to system (2.1) and taking into account Theorem 2.1, we see that (2.1) is (D,T)-stable with probability p if there exists $D_0 \subset D$ such that

$$1 - e^{-\frac{\varphi(D_0)}{\epsilon^2}} > p \quad ,$$

where $\varphi(D_0)$ is the logarithmic first passage probability. Thus, the analysis of (D,T)-stability is equivalent to the calculation of $\varphi(D_0)$ for a given D.

The calculation of $\varphi(D_0)$ according to (2.7) may, however, be difficult. Therefore, simpler sufficient tests for (D,T)-stability and instability are given below:

Theorem 3.1: Assume that A is Hurwitz, (A,C) is disturbable and there exist a matrix $M > 0$ and a number $R_0 > 0$ such that

$$A^T M + MA < 0$$

$$\Gamma \triangleq \min_{y \in \partial D} \|y\| - \frac{R_0}{\sqrt{\lambda_{\min}(M)}} > 0 \quad ,$$

where $\lambda_{\min}(M)$ is the smallest eigenvalue of M and $\|\cdot\|$ is the Eucledean norm of a vector. Then (2.1) is (D,T)-stable with probability $1 - e^{-\alpha/\epsilon^2}$ where

$$\alpha < \frac{\Gamma^2}{2\lambda_{\max}(X(T))} \quad ,$$

$X(t)$ is the covariance matrix defined by (2.6) and $\lambda_{\max}(X(T))$ is the largest eigenvalue of $X(T)$. The corresponding initial set D_0 in this case is:

$$D_0 = \{x \in R^n : \; x^T M x \leq R_0^2\} \quad .$$

Proof: See Appendix 2.

Theorem 3.2: Under the assumption of (A, C) disturbability, system (2.1) is not (D,T)-stable with probability $p = 1 - e^{-\alpha/\epsilon}$ if

$$\frac{\max_{y \in \partial D} \|y\|^2}{2\lambda_{\max}(X(T))} < \alpha \quad .$$

Proof: See Appendix 2.

Thus Theorems 3.1 and 3.2 provide the lower and upper bound for the residence probability in the domain:

$$1 - \exp\left\{ - \frac{\left[\min_{y \in \partial D} \|y\| - \frac{R_0}{\sqrt{\lambda_{\min}(M)}} \right]^2}{2\lambda_{\max}(X(T))\epsilon^2} \right\} < P_{D_0}\{\tau > T\} < 1 - \exp\left\{ - \frac{\max_{y \in \partial D} \|y\|^2}{2\lambda_{\max}(X(T))\epsilon^2} \right\} \quad . \tag{3.3}$$

To illustrate the application of these bounds and the utility of (D,T)-stability, consider an example where two systems have identical LQG performance measures but different (D,T)-stability properties. Specifically, consider system (1.1) with

$$A = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix} , \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \quad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.4}$$

and the initial point x_0 distributed uniformly in

$$D_0 = \{x_0 : \; x_0^T x_0 \le r^2\} \quad .$$

Assume that the goal of control u is to keep $x(t)$ within

$$D = \{x : \; x^T x < R^2\}, \; R > r$$

during the time period T. To find controls that achieve this goal with some accuracy, introduce two performance indecies

$$J_1 = \min_u \frac{1}{2T} \int_0^T \left[\frac{1}{R^2}(x_1^2 + x_2^2) + u^2 \right] dt$$

and

$$J_2 = \min_u \frac{1}{2T} \int_0^T \left[\frac{2}{R^2}(x_1^2 + x_2^2) + 0.0574 \; u^2 \right] dt$$

Minimizing the expected values of these criteria, in the limit as $T \to \infty$, for $R^2 = 2$ we obtain

$$E_{x_0}[J_2] = E_{x_0}[J_2] = 0.0576 \; r^2 \quad .$$

These optimal values are achieved under controls $u_i = K_i x, i = 1, 2$, where

$$K_1 = -\begin{bmatrix} 0.2949 & 0.2247 \end{bmatrix} , \tag{3.5}$$

$$K_2 = -\begin{bmatrix} 2.3627 & 3.2920 \end{bmatrix} , \tag{3.6}$$

respectively. Thus, under gains (3.5) and (3.6) the two closed loop systems have equal LQG performance measures. Next we investigate their (D,T)-stability properties.

Since matrices $A + BK_i$, $i = 1,2$, are Hurwitz and pairs $(A + BK_i,\ C)$, $i = 1,2$, are disturbable, Theorems 3.1 and 3.2 are applicable. Since

$$(A + BK_i)^T + (A + BK_i) < 0 , \quad i = 1,2 ,$$

choose $M = I$, $\min\limits_{y \in \partial D} \|y\| = R$ and $R_0 = r$. The calculations of bounds (3.3) gives:

$$1 - e^{-\frac{1.66(R-r)^2}{\epsilon^2}} < P_{D_0}\{\tau(K_1) > 10\} < 1 - e^{-\frac{1.66R^2}{\epsilon^2}} ,$$

$$1 - e^{-\frac{2.72(R-r)^2}{\epsilon^2}} < P_{D_0}\{\tau(K_2) > 10\} < 1 - e^{-\frac{2.72R^2}{\epsilon^2}} .$$

When r is sufficiently small, the bounds defined by K_1 and K_2 do not overlap and $P_{D_0}\{\tau(K_2) \geq 10\} > P\{\tau(K_1) \geq 10\}$. Analogous situation takes place for T's other than 10. Indeed, for $T = 1$ and $T = 0.1$ we obtain:

$$1 - e^{-\frac{1.6(R-r)^2}{\epsilon^2}} < P_{D_0}\{\tau(K_1) > 1\} < 1 - e^{-\frac{1.6R^2}{\epsilon^2}} ,$$

$$1 - e^{-\frac{2.79(R-r)^2}{\epsilon^2}} < P_{D_0}\{\tau(K_2) > 1\} < 1 - e^{-\frac{2.79R^2}{\epsilon^2}} ;$$

$$1 - e^{-\frac{5.71(R-r)^2}{\epsilon^2}} < P_{D_0}\{\tau(K_1) > 0.1\} < 1 - e^{-\frac{5.71R^2}{\epsilon^2}} \quad ,$$

$$1 - e^{-\frac{7.2(R-r)^2}{\epsilon^2}} < P_{D_0}\{\tau(K_2) > 0.1\} < 1 - e^{-\frac{7.2R^2}{\epsilon^2}} \quad .$$

Hence, in spite of their equivalence in the LQG-sense, the two feedback gains are different in terms of the resulting residence probabilities or (D,T)-stability.

The LQG and the residence probability performance measures could be in direct contradiction with each other. Indeed, continuing example (3.4) with

$$J(d) = \min_u \frac{1}{2T} \int_0^T \left[\frac{d}{R^2}(x_1^2 + x_2^2) + u^2 \right] dt \quad ,$$

it is possible to show that $J(d)$ is an increasing function of d whereas $P_{D_0}\{\tau \leq T\}$ is decreasing. This is illustrated in the Table below:

d	K_d	$E_{x_0}[J(d)]$	$P_{D_0}\{\tau \leq 1\}$
0.5	$-[0.1641\ 0.1180]$	$0.0309\ r^2$	$\left[e^{-\frac{1.54R^2}{\epsilon^2}} \ , \ e^{-\frac{1.54(R-r)^2}{\epsilon^2}}\right]$
5	$-[1.8397\ 2.3166]$	$0.6419\ r^2$	$\left[e^{-\frac{2.53R^2}{\epsilon^2}} \ , \ e^{-\frac{2.53(R-r)^2}{\epsilon^2}}\right]$

4. RESIDENCE PROBABILITY CONTROLLABILITY AND (D,T)-STABILIZABILITY

Consider again system (1.1) with control (2.1). As it follows from Theorem 2.1, if $(A + BK,\ C)$ is disturbable,

$$\lim_{\epsilon \to 0} \epsilon^2 \ln\ P_{D_0}\{\tau(K) \leq T\} = -\varphi(D_0, K) \quad ,$$

where $\varphi(D_0, K)$ is the logarithmic first passage time defined by

$$\varphi(D_0, K) = \min_{x_0 \in [D_0]} \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \frac{1}{2}(y - e^{(A+BK)}x_0)^T X^{-1}(t, K)(y - e^{(A+BK)}x_0) \quad , \qquad (4.1)$$

$$\dot{X}(t, K) = (A + BK)X(t, K) + X(t, K)(A + BK)^T + CC^T, \; X(0, K) = 0 \quad .$$

Note that if $(A, [BC])$ is disturbable, $(A + BK, C)$ is disturbable for almost any K[22]. We would like to choose the feedback law (1.2) so that $\varphi(D_0, K)$ is as large as desired, i.e., the residence probability, which is logarithmically equivalent to $1 - e^{-\frac{\varphi(D_0, K)}{\epsilon^2}}$, is as close to 1 as desired. This may or may not be possible. To characterize the various situations, introduce

Definition 4.1: System (1.1) is said to be *strongly residence probability controllable* (srp-controllable) if for any (D, T) and $\alpha > 0$ there exists $u = Kx$ and $D_0 \subset D$ such that $\varphi(D_0, K) > \alpha$. Otherwise the system is *weakly residence probability controllable.*

The srp-controllability is equivalent to the property of (D, T)-stabilizability:

Definition 4.2: System (1.1) is (D, T)-stabilizable if for any (D, T) and $0 < p < 1$ there exists $u = Kx$ and $D_0 \subset D$ such that

$$P_{D_0}\{\tau > T\} > p \quad .$$

Below, the class of srp-controllable systems is characterized.

Theorem 4.1: Under the assumption of (A, C) disturbability, (1.1) is srp-controllable or, equivalently, (D, T)-stabilizable, if and only if Im $C \subseteq$ Im B.

Proof: See Appendix 3.

If Im $C \not\subseteq$ Im B, there exists an upper bound on the achievable $\varphi(D_0, K)$. This bound is analyzed next.

Consider again (1.1) and assume that B has a full rank. Then there exists a similarity transformation $x = Q\hat{x}$ that transfers (1.1) to the form

$$d\hat{x} = (\hat{A}\hat{x} + \hat{B}u)dt + \epsilon\, \hat{C}dw \tag{4.2}$$

$$\hat{Q}AQ \triangleq \hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix},\ Q^{-1}B = \hat{B} = \begin{bmatrix} I \\ 0 \end{bmatrix},\ Q^{-1}C = \hat{C} = \begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \end{bmatrix} .$$

Since, according to Lemma 2.1, $\varphi(D_0, K) = \hat{\varphi}(\widehat{D}_0, \widehat{K})$, the bounds will be established in terms of the realization (4.2).

Assume (A, B) is controllable and choose

$$u = \widehat{K}_\rho \hat{x},\ \widehat{K}_\rho = -\frac{1}{\rho}\hat{B}^T P_\rho \quad \rho > 0 \ , \tag{4.3}$$

where P_ρ is the positive definite solution of

$$\widehat{A}^T P_\rho + P_\rho \hat{A} + I - \frac{1}{\rho} P_\rho \hat{B}\hat{B}^T P_\rho = 0 \ . \tag{4.4}$$

Let P_{22} be the positive definite solution of

$$\hat{A}_{11}P_{22} + P_{22}\hat{A}_{11} + I - P_{22}\hat{A}_{21}\hat{A}_{21}^T P_{22} = 0 \tag{4.5}$$

and

$$\tilde{A} = -\hat{A}_{21}\hat{A}_{21}^T P_{22} + \hat{A}_{22} \ . \tag{4.6}$$

Let α and β be positive numbers that satisfy the inequality:

$$(\|P_{22}\hat{A}_{21}\hat{A}_{21}^T P_{22}\|_2 + 1)\|e^{\widetilde{A}t}\|_2^2 \leq \alpha e^{-2\beta t} \quad , \tag{4.7}$$

where $\|\cdot\|_2$ is the induced l_2-norm of a matrix. Finally let $M > 0$ and $R_0 > 0$ be a matrix and a number, respectively, satisfying

$$\lim_{\rho \to 0}[(\hat{A} + \hat{B}\widehat{K}_\rho)^T M + M(\hat{A} + \hat{B}\widehat{K}_\rho)] < 0 \tag{4.8}$$

$$\Gamma = \min_{y \in \partial D} \|y\| - \frac{R_0}{\sqrt{\lambda_{\min}(M)}} > 0 \ . \tag{4.9}$$

Theorem 4.2: Under the assumption of controllability of (A, B) and disturbability of (A, C) and for all $T > 0$ such that

$$1 - \alpha e^{-2\beta T} > 0 \quad , \tag{4.10}$$

the maximal achievable logarithmic first passage time, $\max_\rho \hat{\varphi}(\widehat{D}_0, \widehat{K}_\rho)$, is bounded as follows:

$$\frac{\Gamma^2}{2\lambda^{**}} \leq \max_\rho \hat{\varphi}(\widehat{D}_0, \widehat{K}_\rho) = \lim_{\rho \to 0} \hat{\varphi}(\widehat{D}_0, \widehat{K}_\rho) \leq \frac{\max_{y \in \partial D} \|y\|^2}{2\lambda^*(1 - \alpha e^{-2\beta T})} \quad , \tag{4.11}$$

where

$$\lambda^* = \frac{\lambda_{\max}(\hat{C}_2 \hat{C}_2^T)}{2[\lambda_{\max}(\hat{A}_{21}\hat{A}_{21}^T + \hat{A}_{22}\hat{A}_{22}^T)]^{1/2}} \tag{4.12}$$

$$\lambda^{**} = \mathrm{Tr}\hat{C}_2^T P_{22} \hat{C}_2 \tag{4.13}$$

and Γ, M, α and β are defined by (4.9), (4.8) and (4.7), respectively.

Proof: See Appendix 3.

Note that in the srp-controllability case $\hat{C}_2 = 0$, $\lambda^* = \lambda^{**} = 0$ and, therefore,

$$\lim_{\rho \to 0} \varphi(D_0, K_\rho) = \infty \quad .$$

As it has been pointed out in Section 1, the residence probability control problem has a more complex nature than residence time control. This complexity manifests itself through the fact that, unlike the residence time (see [8], formula (3.6)), the bounds on the maximal achievable residence probability depend on the desired period of operation T and, more importantly, on the size of the initial, "lock in" set D_0.

To illustrate the bounds of Theorem 4.3, consider an example of the roll attitude control problem in a missile disturbed by a random torque [23]. The dynamics of the system are described as

$$\begin{bmatrix} \dot{\delta} \\ \dot{\omega} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 10 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \delta \\ \omega \\ \varphi \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u + \epsilon \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \dot{w} \tag{4.14}$$

where δ is the aileron deflection, ω is the roll angular velocity, φ is the roll angle, u is control of aileron actuators and $\dot{w}$ is white noise. Note that (4.14) is in the form (4.2) with $\hat{C}_2 \neq 0$, i.e., the system under consideration is wrp-controllable and $\varphi(D_0, K)$ is bounded. To evaluate this bound, assume, for simplicity, that D is a ball with radius R and calculate

$$\lambda^* = 0.04975 \ , \quad \lambda^{**} = 0.1 \ , \quad \alpha = 12 \ , \quad \beta = 2 \quad .$$

Choosing M as

$$\begin{bmatrix} 0.25 & 0.25 & 0.25 \\ 0.25 & 0.35 & 0.35 \\ 0.25 & 0.35 & 1.35 \end{bmatrix} \tag{4.15}$$

and varifying that (4.8) holds, we finally obtain:

$$5\left(R - \frac{R_0}{\sqrt{0.042}}\right)^2 \leq \lim_{p\to 0} \varphi(D_0, K_p) \leq \frac{10.05R^2}{1 - 12e^{-2T}},\ T > \frac{\ln 12}{2} \ .$$

Thus, there is no linear controller (4.3) for missile (4.14) that keeps the states in the ball of radius R during interval T with probability $p > 1 - \exp\left\{-\frac{10.05R^2}{(1-12e^{-2T})\epsilon^2}\right\}$. On the other hand, there exists a linear state feedback that accomplishes this task with probability $p \leq 1 - \exp\left\{-\frac{5}{\epsilon^2}\left(R - \frac{R_0}{\sqrt{0.042}}\right)\right\}$, provided that at $t = 0$ the states are locked into the initial set $D_0 = \{x : \ x^T M x \leq R_0\}$, where M is given by (4.15).

5. CONCLUSIONS

1. The residence probability control gives a stronger reformulation of the aiming control problem than the residence time control. However, the resulting control problem is also more complex: the performance depends on the size of the initial, "lock in", domain and on the operating period.

2. The (D,T)-stability with probability p is a useful tool for characterization of the performance of stochastic systems with no equilibrium points. The performance in terms of (D,T)-stability may be contradictory to the performance

in terms of LQG criterion.

3. The residence probability of a controlled linear system with additive white noise can be modified in any desired manner, for instance, made as close to 1 as desired, if and only if the range space of the noise matrix is included in the range space of the control matrix. Otherwise, the achievable residence probability is bounded away from 1 and estimates of this bound are characterized herein.

APPENDIX 1

Proof of Theorem 3.1:

Consider

$$dx = Axdt + \epsilon\, Cdw, \; x(0) = x_0 \quad , \tag{A1.1}$$

where $x \in R^n$, $0 < \epsilon \ll 1$ and w is a standard r-dimensional Brownian motion. Let $U(t)$ be an absolutely continuous function in R^r and define $F_{x_0}(U)$ as follows:

$$F_{x_0}(U) \triangleq \phi = x_0 + \int_0^T A\phi dt + CU \quad . \tag{A1.2}$$

The mapping $\mathrm{F}_{x_0}(U)$ is continuous with respect to U and, assuming without loss of generality that C has a full row rank, one-to-one. Therefore, as it follows from Theorems 3.1 (Chapter 3) and 1.1 (Chapter 4) of [7], the action functional for (A1.1) is:

$$S_{0T}(\phi) \triangleq S_{0T}(u) = \begin{cases} \infty\,, & \text{if } F_{x_0}^{-1}(\phi) \text{ is empty} \\ \frac{1}{2}\int_0^T u^T u dt, \; \dot{\phi} = A\phi + Cu, \; \phi(0) = x_0, & \text{otherwise,} \end{cases} \tag{A1.3}$$

where $u(t) = \dot{U}(t)$.

Define

$$\tau_{x_0} \triangleq \min\{t : \; x(t) \in \partial D | x_0 \in D\} \quad ,$$

where $D \subset R^n$ is an open bounded set in R^n with 0 in its interior and smooth

boundary ∂D and $x(t)$ is the trajectory of (A1.1). Introduce also

$$\overline{H}_D(T, x_0) \triangleq \{\phi \in C_{0T}(R^n) : \ \phi(0) = x_0 \in D, \ \phi(s) \notin D \text{ for some } s \in [0, T]\} \tag{A1.4}$$

$$\overline{U}_D(T, x_0) \triangleq \{u \subset C_{0T}(R^n) : \ \phi \in \overline{H}_D(T, x_0), \ \dot{\phi} = A\phi + Cu, \ \phi(0) = x_0\} \tag{A1.5}$$

It follows from Theorems 1.1 and 1.2 (Chapter 4) of [7] that if

$$\inf_{u \in \overline{U}_D(T, x_0)} \{S_{0T}(u) : \ \phi \in [H_D(T, x_0)]\} = \inf_{u \in \overline{U}_D(T, x_0)} \{S_{0T}(u) : \ \phi \in \overline{H}_D(T, x_0)\} \ , \tag{A1.6}$$

then uniformly with respect to all $x_0 \in R^n$,

$$\lim_{\epsilon \to 0} \epsilon^2 \ln \text{Prob}\{\tau_{x_0} \le T\} = - \min_{u \in \overline{U}_D(T, x_0)} S_{0T}(u) \ . \tag{A1.7}$$

To prove that (A1.6) holds assume that the minimum of $S_{0T}(u)$ on set $\overline{U}_D(T, x_0)$ is attained at function u^*. Denote the corresponding $\phi(t)$ as ϕ^* and assume that ϕ^* reaches ∂D at t^*, i.e., $\phi^*(t^*) \in \partial D$. Define the neighborhood of $\phi^*(t^*)$ as follows:

$$N_{\phi^*(t^*)}(\delta) \triangleq \{x \in R^n : \ |\phi^*(t^*) - x| < \delta, \ \delta > 0\} \ .$$

For any $\delta > 0$, choose $x^\delta \in N_{\phi^*(t^*)}$ which is not contained in D. Due to the assumption that (A, C) is disturbable there exists u^δ such that $\phi(u^\delta(s)) = x^\delta$ for some $s \in [0, T]$. Therefore, $\phi(u^\delta)$ is in the interior of $\overline{H}_D(T, x_0)$. Since $S_{0T}(u)$ is differentiable and

$$\lim_{\delta \to 0} S_{0T}(u^\delta) = S_{0T}(u^*) \ ,$$

this implies that condition (A1.6) is satisfied, and (A1.7) holds.

To solve the minimization problem of (A1.7) we observe that, as it follows from [7], p. 107,

$$\min_{u \in \overline{U}_D(T,x_0)} S_{0T}(u) = \min_{\substack{y \in \partial D \\ 0 \le t \le T}} V(t,\ x_0, y) \tag{A1.8}$$

where

$$V(t, x_0, y) = \min_u \frac{1}{2} \int_0^t u^T u dt \tag{A1.9}$$

$$\dot{\phi} = A\phi + Cu,\ \phi(0) = x_0,\ \phi(t) = y\ \ . \tag{A1.10}$$

Problem (A1.9), (A1.10) can be solved using a standard variational approach. The necessary conditions of optimality,

$$\begin{aligned} \dot{\phi}^* &= A\phi^* + Cu^*\ ,\ \phi^*(0) = x_0\ ,\ \phi^*(t) = y \\ \dot{p}^* &= -A^T p^* \\ u^* &= -C^T p^*\ \ , \end{aligned}$$

result in

$$u^*(\tau) = -C^T e^{-A^T(t-\tau)} X^{-1}(t)(y - e^{At}x_0)$$

and, hence,

$$V(t, x_0, y) = \frac{1}{2}(y - e^{At}x_0)^T X^{-1}(t)(y - e^{At}x_0)\ \ ,$$

where

$$\dot{X} = AX + XA^T + CC^T,\ \ X(0) = 0\ \ .$$

Thus, according to (A1.8),

$$\min_{u\in\overline{U}_D(T,x_0)} S_{0T}(u) = \min_{\substack{y\in\partial D\\ 0\le t\le T}} \frac{1}{2}(y - e^{At}x_0)^T X^{-1}(t)(y - e^{At}x_0)$$

which, together with (A1.7), defines the distribution of the first passage time of a trajectory originating at x_0.

To complete the proof consider

$$P_{D_0}\{\tau \le T\} = \max_{x_0\in[D_0]} \mathrm{Prob}\{\tau_{x_0} \le T\} \quad ,$$

where $D_0 \subset D$ is an open set containing 0 with a smooth boundary ∂D_0. Taking into account the continuity of ln $\mathrm{Prob}\{\tau_{x_0} \le T\}$, the uniformity of the limit in (A1.7), and the compactness of $[D_0]$,

$$\begin{aligned}
&\lim_{\epsilon\to 0} \epsilon^2 \ln P_{D_0}\{\tau_{x_0} \le T\} = \lim_{\epsilon\to 0} \epsilon^2 \ln \max_{x_0\in[D_0]} \mathrm{Prob}\{\tau_{x_0} \le T\} \\
= \ &\lim_{\epsilon\to 0} \epsilon^2 \max_{x_0\in[D_0]} \ln \mathrm{Prob}\{\tau_{x_0} \le T\} = \max_{x_0\in[D_0]} \lim_{\epsilon\to 0} \epsilon^2 \ln \mathrm{Prob}\{\tau_{x_0} \le T\} \\
= \ &\max_{x_0\in[D_0]} \left[- \min_{\substack{y\in\partial D\\ 0\le t\le T}} \frac{1}{2}(y - e^{At}x_0)^T X^{-1}(t)(y - e^{At}x_0) \right] \\
= \ &- \min_{x_0\in[D_0]} \min_{\substack{y\in\partial D\\ 0\le t\le T}} \frac{1}{2}(y - e^{At}x_0)^T X^{-1}(t)(y - e^{At}x_0) \quad .
\end{aligned}$$

Q.E.D.

Proof of Lemma 2.1: Since $X(t) = Q\widehat{X}(t)Q^T$,

$$\begin{aligned}
&(\hat{y} - e^{\hat{A}t}\hat{x}_0)^T \widehat{X}^{-1}(t)(\hat{y} - e^{\hat{A}t}\hat{x}_0) \\
= \ &(Q^{-1}y - Q^{-1}e^{At}QQ^{-1}x_0)^T Q^T X^{-1}(t) Q (Q^{-1}y - Q^{-1}e^{At}QQ^{-1}x_0) \\
= \ &(y - e^{At}x_0)^T X^{-1}(t)(y - e^{At}x_0) \quad .
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\hat{\varphi}(\widehat{D}_0) &= \min_{\hat{x}_0 \in [\hat{D}_0]} \min_{\substack{\hat{y} \in \partial \hat{D} \\ 0 \le t \le T}} \frac{1}{2} (\hat{y} - e^{\hat{A}t} \hat{x}_0)^T \widehat{X}^{-1}(t) (\hat{y} - e^{\hat{A}t} \hat{x}_0) \\
&= \min_{x_0 \in [D_0]} \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \frac{1}{2} (y - e^{At} x_0)^T X^{-1}(t) (y - e^{At} x_0) = \varphi(D_0) \quad .
\end{aligned}
$$

Q.E.D.

APPENDIX 2

Proof of Theorem 3.1: Observe that

$$
\begin{aligned}
\varphi(D_0) &\geq \min_{x_0 \in [D_0]} \min_{\substack{y \in \partial D \\ 0 \leq t \leq T}} \frac{1}{2} \lambda_{\min}(X^{-1}(t)) \|y - e^{At} x_0\|^2 \\
&\geq \frac{\min_{x_0 \in [D_0]} \min_{\substack{y \in \partial D \\ 0 \leq t \leq T}} \|y - e^{At} x_0\|^2}{2\lambda_{\max}(X(T))} \quad .
\end{aligned}
$$

Since A is Hurwitz, there exists $M > 0$ which satisfies

$$A^T M + MA < 0 \quad .$$

Then $D_0 = \{x \in R^n : \; x^T M x \leq R_0^2\}$ is an invariant set for (2.1) with $\epsilon = 0$. If

$$\Gamma \triangleq \min_{y \in \partial D} \|y\| - \frac{R_0}{\sqrt{\lambda_{\min}(M)}} > 0 \quad ,$$

this invariant set is contained in D. Therefore,

$$\varphi(D_0) \geq \frac{\min_{x_0 \in [D_0]} \min_{\substack{y \in \partial D \\ 0 \leq t \leq T}} \|y - e^{At} x_0\|^2}{2\lambda_{\max}(X(t))} \geq \frac{\left(\min_{y \in \partial D} \|y\| - \frac{R_0}{\sqrt{\lambda_{\min}(M)}}\right)^2}{2\lambda_{\max}(X(t))} \triangleq \Phi_1 \quad .$$

This implies that (2.1) is (D, T)-stable with probability $1 - e^{-\alpha/\epsilon^2}$ where

$$0 < \alpha < \Phi_1 \quad .$$

Q.E.D.

Proof of Theorem 3.2:

$$
\begin{aligned}
\varphi(D_0) &\leq \min_{\substack{y\in\partial D\\ 0\leq t\leq T}} \frac{1}{2} y^T X^{-1}(t) y \\
&\leq \min_{\substack{y\in\partial B\\ 0\leq t\leq T}} \frac{1}{2} y^T X^{-1}(t) y = \min_{0\leq t\leq T} \frac{1}{2}\lambda_{\min}(X^{-1}(t))R^2 \\
&= \frac{R^2}{2\lambda_{\max}(X(T))} \triangleq \Phi_2 \ ,
\end{aligned}
$$

where $B = \{x \in R^n : \ \|x\| \leq R\}$ and $R = \max_{y\in\partial D} \|y\|$. This implies that (2.1) is not (D,T)-stable with probability $1 - e^{-\alpha/\epsilon^2}$, where $\alpha > \Phi_2$.

Q.E.D.

APPENDIX 3

Proof of Theorem 4.1: Sufficiency: If $\mathrm{Im}\, C \subseteq \mathrm{Im}\, B$, the disturbability of (A,C) guarantees the controllability of (A,B). Then, as it was shown in [8], there exist a sequence $\{K_\alpha\}$ such that $(A+BK_\alpha)$ is Hurwitz for $\forall\alpha \in [1,\infty)$, a set D_0 which is invariant set for $\dot{x} = (A+BK_\alpha)x$ for $\forall\alpha$ and, in addition,

$$\lim_{\alpha\to\infty} X_\infty(K_\alpha) = 0 \ ,$$

where

$$(A+BK_\alpha)X_\infty(K_\alpha) + X_\infty(K_\alpha(A+BK_\alpha)^T + CC^T = 0 \ .$$

Since $X(T,K_\alpha) \le X_\infty(K_\alpha)$, $\forall T \ge 0$

$$\lim_{\alpha\to\infty} X(T,K_\alpha) = 0 \ , \quad \forall T \ge 0 \ ,$$

where

$$\dot{X}(t,K_\alpha) = (A+BK_\alpha)X_\infty(K_\alpha) + X_\infty(K_\alpha)(A+BK_\alpha)^T + CC^T \ .$$

Therefore, by using Theorem 3.1,

$$\lim_{\alpha\to\infty} \varphi(D_0,K_\alpha) = \infty \ ,$$

which implies that (1.1) is srp-controllable .

Necessity: Assume that (1.1) is srp-controllable. Then there exists a sequence $\{K_\alpha\}$ such that $\lim_{\alpha\to\infty} \varphi(D_0,K_\alpha) = \infty$, which implies that $\lim_{\alpha\to\infty} \lambda_{\max}(X(T,K_\alpha)) =$

0 or $\lim_{\alpha\to\infty} TrX(T, K_\alpha) = 0$. Then by Fatau's lemma,

$$0 = \lim_{\alpha\to\infty} TrX(T, K_\alpha) = \lim_{\alpha\to\infty} Tr \int_0^T e^{(A+BK_\alpha)t} CC^T e^{(A+BK_\alpha)^T t} dt$$

$$\geq \int_0^T \lim_{\alpha\to\infty} \inf_t Tr(e^{(A+BK_\alpha)t} CC^T e^{(A+BK_\alpha)^T t}) dt \quad . \qquad \text{(A3.1)}$$

Since $e^{(A+BK_\alpha)t} CC^T e^{(A+BK_\alpha)^T t} \geq 0$, $\forall\, t \in [0, T]$ and α, (A3.1) gives:

$$\lim_{\alpha\to\infty} \inf Tr(e^{(A+BK_\alpha)t} CC^T e^{(A+BK_\alpha)^T t}) = 0$$

for almost all $t \in [0, T]$. Therefore, there exists a subsequence $\{K_\beta\}$ of $\{K_\alpha\}$ such that

$$\lim_{\beta\to\infty} Tr(e^{(A+BK_\beta)t} CC^T e^{(A+BK_\beta)^T t}) = 0 \qquad \text{(A3.2)}$$

for almost all $t \in [0, T]$. Noting that $X(t, K_\beta)$ obeys the equation

$$(A + BK_\beta)X(t, K_\beta) + X(t, K_\beta)(A + BK_\beta)^T + CC^T$$

$$= e^{(A+BK_\beta)t} CC^T e^{(A+BK_\beta)^T t} \quad , \quad \forall\, t \quad , \qquad \text{(A3.3)}$$

from (A3.2) and (A3.3) it follows that

$$\lim_{\beta\to\infty} [(A + BK_\beta)X(t, K_\beta) + X(t, K_\beta)(A + BK_\beta)^T + CC^T] = 0 \qquad \text{(A3.4)}$$

for almost all $t \in [0, T]$. Since we know that $X(t, K_\beta) \to 0$ as $\beta \to \infty$, (A3.4) implies that

$$\lim_{\beta\to\infty} [BK_\beta X(t, K_\beta) + X(t, K_\beta) K_\beta^T B^T] + CC^T = 0 \qquad \text{(A3.5)}$$

for almost all $t \in [0,T]$. Therefore, there exists $Q(t)$ such that

$$BQ(t) + Q^T(t)B^T + CC^T = 0 \tag{A3.6}$$

which is possible only if Im $C \subseteq$ Im B.

Q.E.D.

Proof of Theorem 4.2: From Theorem 3.2 we know that

$$\varphi(D_0, K_\rho) \leq \frac{R^2}{\lambda_{max}(X(T, K_\rho)} \ , \tag{A3.7}$$

where $R^2 = \max_{y \in \partial D} \|y\|^2$. Ommiting, for the sake of brevity argument K_ρ of function $X(T, K_\rho)$ and taking into account that

$$X(T) = X_\infty - e^{(A+BK_\rho)T} X_\infty e^{(A+BK_\rho)^T T} \ ,$$

where $X_\infty = \lim_{T \to \infty} X(T)$, the denominator of (A3.7) can be bounded as

$$\lambda_{max}(X(T)) \geq \lambda_{max}(X_\infty)(1 - \|e^{(A+BK_\rho)T}\|_2^2) \ . \tag{A3.8}$$

In [8] it has been shown that $\lambda_{max}(X_\infty) \geq \lambda^* > 0$ defined in (4.12). Therefore, to specify estimate (A3.7), $\lim_{\rho \to 0} \|e^{(A+BK_\rho)T}\|_2^2$ should be evaluated.

With this in mind, consider the realization (4.2) with control (4.3) (again, for simplicity of notations we drop the sign $\wedge$ in all appropriate symbols):

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} - \frac{1}{\rho}P_{p11} & A_{12} - \frac{1}{\rho}P_{p12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{A3.9}$$

where $P_{\rho ij}$ are defined by the positive definite solution of (4.4) represented as follows:

$$P_\rho = \begin{bmatrix} P_{\rho 11} & P_{\rho 12} \\ P_{\rho 12}^T & P_{\rho 22} \end{bmatrix} .$$

For ρ sufficiently small, (A3.9) can be analyzed using the singular perturbation approach [24]. This results in

$$\overline{P}_{11}x_1(t) = -\overline{P}_{12}x_2(t), \quad \forall\, t > 0 \tag{A3.10}$$

where

$$\overline{P}_{11} = \lim_{\rho\to 0}\frac{P_{\rho 11}}{\sqrt{\rho}} \quad , \quad \overline{P}_{12} = \lim_{\rho\to 0}\frac{P_{\rho 12}}{\sqrt{\rho}} \quad .$$

In addition, since $\lim\limits_{\rho\to 0} P_\rho < \infty$, it follows from (4.4) that

$$\lim_{\rho\to 0} P_\rho BB^T P_p = \lim_{\rho\to 0}\begin{bmatrix} P_{\rho 11}^2 & P_{\rho 11}P_{\rho 12} \\ P_{\rho 12}^T P_{\rho 11} & P_{\rho 12}^T P_{\rho 12} \end{bmatrix} = 0$$

and, therefore, $P_{\rho 11} = P_{\rho 12} = 0$ as $\rho \to 0$. Denote

$$P_{22} \triangleq \lim_{\rho\to 0} P_{\rho 22} \quad ,$$

$$\begin{bmatrix} \overline{P}_{11}^2 & \overline{P}_{11}\overline{P}_{12} \\ \overline{P}_{12}^T\overline{P}_{11} & \overline{P}_{12}^T\overline{P}_{12} \end{bmatrix} \triangleq \lim_{\rho\to 0}\frac{1}{\rho}\begin{bmatrix} P_{\rho 11}^2 & P_{\rho 11}P_{\rho 12} \\ P_{\rho 12}^T P_{\rho 11} & P_{\rho 12}^T P_{\rho 12} \end{bmatrix}$$

Then from the above considerations it follows that

$$\lim_{\rho\to 0}[A^T P_\rho + P_\rho A + I - \frac{1}{\rho}P_p BB^T P_\rho]$$
$$= \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & P_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & P_{22} \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + I$$

$$-\begin{bmatrix} \overline{P}_{11}^2 & \overline{P}_{11}\overline{P}_{12} \\ \overline{P}_{12}^T\overline{P}_{11} & \overline{P}_{12}^T\overline{P}_{12} \end{bmatrix} = 0 \ , \tag{A3.11}$$

i.e.,

$$I - \overline{P}_{11}^2 = 0 \ , \tag{A3.12}$$

$$A_{21}^T P_{22} - \overline{P}_{11}\overline{P}_{12} = 0 \ , \tag{A3.13}$$

$$A_{22}^T P_{22} + P_{22}A_{22} + I - \overline{P}_{12}^T\overline{P}_{12} = 0 \ . \tag{A3.14}$$

Since $P_\rho > 0$, from (A3.12) - (A3.14), respectively, we obtain

$$\overline{P}_{11} = I \ ,$$

$$\overline{P}_{12} = A_{21}^T P_{22}$$

$$A_{22}^T P_{22} + P_{22}A_{22} + I - P_{22}A_{21}A_{21}^T P_{22} = 0 \ ,$$

Therefore, from (A3.10) and (A3.9)

$$x_1(t) = -A_{21}^T P_{22} x_2(t), \quad t > 0$$

and

$$\dot{x}_2(t) = \tilde{A} x_2(t), \quad t > 0$$

where

$$\tilde{A} = -A_{21}A_{21}^T P_{22} + A_{22}$$

is a Hurwitz matrix. Thus,

$$\lim_{\rho\to 0} e^{(A+BK_\rho)t} = \begin{bmatrix} 0 & -A_{21}^T P_{22} e^{\widetilde{A}t} \\ 0 & e^{\widetilde{A}t} \end{bmatrix}$$

and

$$\begin{aligned} \|\lim_{\rho\to 0} e^{(A+BK_\rho)t}\|_2 &= \|e^{\widetilde{A}^T t} e^{\widetilde{A}t} + e^{\widetilde{A}^T t} P_{22} A_{21} A_{21}^T P_{22} e^{\widetilde{A}t}\|_2 \\ &\leq (1 + \|P_{22} A_{21} A_{21}^T P_{22}\|_2)\|e^{\widetilde{A}t}\|_2^2 \quad \forall t > 0 \ . \end{aligned} \tag{A3.15}$$

Therefore, for $\alpha > 0$ and $\beta > 0$ satisfying (4.7), from (A3.8) and (A3.15),

$$\lim_{\rho\to 0} \lambda_{\max}(X(T, K_\rho)) \geq \lambda^*(1 - \alpha e^{-2\beta T})$$

and

$$\lim_{\rho\to 0} \varphi(D_0, K_\rho) \leq \frac{R^2}{2\lambda^*(1 - \alpha e^{-2\beta T})} \quad \text{if} \quad 1 - \alpha e^{-2\beta T} > 0 \ .$$

This provides the upper bound for $\varphi(D_0, K_\rho)$.

To obtain the lower bound, we write

$$\lambda_{\max}(X(T, K_\rho)) \leq \lambda_{\max}(X_\infty(K_\rho)) \leq \mathrm{Tr}\ X_\infty(K_\rho) \ .$$

Then, by [25],

$$\lim_{\rho\to 0} \mathrm{Tr}\ X_\infty(K_\rho) = \lim_{\rho\to 0} \mathrm{Tr}\ C^T P_\rho C = \mathrm{Tr}\ C_2^T P_{22} C_2 \triangleq \lambda^{**} \ .$$

Hence, by Theorem 3.1,

$$\lim_{\rho\to 0} \varphi(D_0, K_\rho) \geq \frac{\left(R - \frac{R_0}{\sqrt{\lambda_{\min}(M)}}\right)^2}{2\lambda^{**}} \ .$$

Q.E.D.

APPENDIX 4

Consider the closed loop system (1.1), (1.2) with

$$(A+BK) = A_c(a,b) = \begin{bmatrix} 0 & 1 \\ -\frac{a}{b} & -b \end{bmatrix} \ , \quad C = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ . \tag{A4.1}$$

If $e^{A_c(a,b)t}x_0 \in D$ for $\forall\ t \geq 0$, the average residence time, $E[\tau_{x_0}]$, as estimated in [8], is

$$\lim_{\epsilon \to 0} \epsilon^2 \ln E[\tau_{x_0}] = \frac{1}{2} \min_{y \in \partial D} y^T X_\infty^{-1} y \geq \frac{\min\limits_{y \in \partial D} y^T y}{2\lambda_{\max}(X_\infty)} \ , \tag{A4.2}$$

where

$$A_c X_\infty + X_\infty A_c^T + CC^T = 0 \ .$$

Since in the case of (A4.1)

$$X_\infty = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} \ ,$$

from (A4.2) it follows that

$$E[\tau_{x_0}] \geq \exp\left\{ \frac{b \min\limits_{y \in \partial D} y^T y}{\epsilon^2} \right\} \ \text{if } b \leq a \ .$$

Therefore, $E[\tau_{x_0}]$ can be made as large as desired by an appropriate choice of b.

The residence probability, as it follows from Theorem 2.1 is:

$$\lim_{\epsilon \to 0} \epsilon^2 \ln \ \text{Prob}\{\tau_{x_0} \leq T\} = -\frac{1}{2} \min_{\substack{y \in \partial D \\ 0 \leq t \leq T}} (y - e^{A_c(a,b)t}x_0)^T X^{-1}(t)(y - e^{A_c(a,b)t}x_0)$$

$$\geq -\frac{1}{2} \min_{\substack{y\in\partial D \\ 0\leq t\leq T}} \frac{\|y - e^{A_c(a,b)}x_0\|^2}{\lambda_{\min}(X(t))} \quad ,$$

where

$$\dot{X}(t) = A_c(a,b)X(t) + X(t)A_c^T(a,b) + CC^T, \; X(0) > 0 \quad .$$

If $b^3 \leq 4a$,

$$e^{A_c(a,b)t} = \begin{bmatrix} \cos\delta t + \frac{\sigma}{\delta}\sin\delta t & \frac{1}{\delta}\sin\delta t \\ -\frac{\sigma^2+\delta^2}{\delta}\sin\delta t & \cos\delta t - \frac{\sigma}{\delta}\sin\delta t \end{bmatrix}$$

where $\sigma = b/2$ and $\delta = \frac{1}{2}\sqrt{\frac{4a}{b} - b^2}$. Therefore,

$$\lim_{a\to\infty} \sup_{t\geq 0} \sup_{\|x_0\|\leq 1} \|e^{A_c(a,b)t}\|_2 = \infty \quad .$$

Thus, for any open $D_0 \subset D$ with 0 in its interior, $T > 0$, and $b > 0$ there exists $\hat{x}_0$ and $\hat{a}$ such that $e^{A_c(\hat{a},b)t}\hat{x}_0$ is arbitrarily close to ∂D at some $t \in [0,\ T]$. Then, as it follows from (A4.3)

$$\text{Prob}\{\tau_{x_0} \leq T\} \geq \exp - \left\{ \frac{\min\limits_{\substack{y\in\partial D \\ 0\leq t\leq T}} \|y - e^{A_c(\hat{a},b)}\hat{x}_0\|}{2\lambda_{\min}(X(t))} \right\}$$

is as close to 1 as desired and $\text{Prob}\{\tau_{x_0} > T\}$ small as desired.

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APPENDIX 1

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THEORY OF AIMING CONTROL FOR LINEAR STOCHASTIC SYSTEMS

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Abstract. The problem of aiming control is formulated and solved for linear time-invariant systems perturbed by small additive white noise. Both state feedback and dynamic output feedback laws are considered. In the case of dynamic output feedback, noiseless and noisy measurements are analyzed. In the noiseless measurement case, it is shown that the fundamental bounds on the achievable precision of aiming may or may not be finite depending on the nonminimum phase zeros and the invertibility of the system. In the noisy measurement case the achievable precision of aiming is shown to be always bounded. Aiming controller design techniques that result in controllers compatible with the bounds are developed. The approach is based on the asymptotic large deviations theory.

Keywords. Stochastic control; pointing control; large deviations; first passage times.

INTRODUCTION

Given a dynamical system with states $x(t)$, control $u(t)$, output $y(t)$ and disturbances $\xi(t)$, assume its desired behaviour is specified by a pair $\{\Psi, \tau\}$, where Ψ is the domain to which the output $y(t)$ should be confined and τ is the period of the confinement, i.e., $y(t) \in \Psi, t \in [0, \tau]$. For a given pair $\{\Psi, \tau\}$ we formulate the pair of *aiming control* as the problem of choosing a feedback control law u, so a to force the output $y(t)$ to remain, at least on average, in Ψ during period τ, in spite of the disturbances $\xi(t)$ that are acting on the system.

Design specifications of this form arise in numerous practical control problem, e.g., telescope pointing, beam pointing, missle guidance and airplane landing. These and other examples are discussed in Meerkov and Runolfsson (1988).

Existing control theory does not offer tools for a direct solution of the aiming control problem described above. In this paper a theory of aiming control of linear systems perturbed by small additive white noise is presented. The approach is based on the modern asymptotic large deviations theory.

The theory described in this paper is an extension of a theory developed earlier by the authors in Meerkov and Runolfsson (1988, 1989) and Runolfsson (1989). In Meerkov and Runolfsson (1988, 1989) the theory was developed for linear systems with small white noise perturbations and state feedback control laws. In Runolfsson (1989) systems with noisy measurements and direct (static) output feedback control laws were considered. In the present paper the genearal case of dynamic output feedback with noisy (and noise-free) measurements is considerd.

As in the earlier papers (Meerkov and Runolfsson (1988, 1989)) we divide all stabilizable linear systems into two classes, weakly and strongly residence time controllable. Roughly speaking, the system is weakly residence time controllable (*wrt*-controllable) if there exists a $0 < \tau^* < \infty$ such that the aiming control specifications can be satisfied by a choice of $u = u(\tau, x)$ for all $\tau < \tau^*$ but not for any $\tau > \tau^*$. The system is stongly residence time controllable (*srt*-controllable) if $\tau^* = \infty$.

The basic results, for linear time invariant system perturbed by small additive white noise, derived in the paper can be summarized as follows:

1. A system that has perfect (i.e., noise-free) measurements is *srt*-controllable if and only if the system is minimum phase and invertible in an appropriate sense.
2. Systems with noisy measurements are never *srt*-controllable. Thus, the effect of the measurement noise is more detrimental on the aiming ability of a system than the input noise.
3. The observer gain that ensures the best precision of aiming is the Kalman filter gain. Thus, the Kalman filter is optimal not only with respect to the standard performance measure (the mean square estimation error) but also from the point of view of the residence time.
4. The controller gain that achieves the best precision of aiming depends on the optimal value of the observer gain. Thus, although the separation principle does not take place, the situation can be characterized as a semi-separation: the optimal observations do not depend

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on the optimal control but the optimal control does depend on the optimal observations.

The paper is organized as follows: In Section 2 we present some mathematical preliminaries and give a presice formulation of the control problem. In Section 3 an analysis of residence time controllability properties is given and in Section 4 design techniques are considered. An illustrative example is given in Section 5 and conclusions can be found in Section 6. Due to space limitations, all proofs are omitted and are available from the authors upon request.

PRELIMINARIES AND PROBLEM FORMULATION

Consider the following linear Ito system

$$\begin{aligned} dx &= Axdt + \epsilon C dw \\ y &= Dx \end{aligned} \tag{1}$$

where $x \in R^n, y \in R^p, w(t)$ is a standard r-dimensional Brownian motion and $0 < \epsilon \ll 1$. Let $\Psi \subset R^p$ be a bounded domain with the origin in its interior and a smooth boundary $\partial\Psi$. Define

$$\Omega_0 = \{x \in R^n | y = Dx \in \Psi\}, \tag{2}$$

$$\Omega = \left\{x \in R^n | De^{At}x \in \Psi, t \geq 0\right\}. \tag{3}$$

Assume that $x(0) = x_0 \in \Omega_0$ and introduce the first passage time

$$\tau^\epsilon(x_0) = \inf\{t \geq 0 | y(t, x_0) \in \Psi\}, \tag{4}$$

where $y(t, x_0)$ is the solution of (1) with initial condition x_0. The following theorem was proven in Meerkov and Runolfsson (1989) (see also Freidlin and Wentzell (1984)).

Theorem 1: Suppose A is Hurwitz and (A, C) is disturbable, i.e., $\mathrm{rank}[C \quad AC \quad \cdots \quad A^{n-1}C] = n$. Then uniformly for all x_0 belonging to compact subsets of Ω we have

$$\lim_{\epsilon \to 0} \epsilon^2 \ln \bar{\tau}^\epsilon(x_0) = \hat{\mu}, \tag{5}$$

where $\bar{\tau}^\epsilon(x_0) = E_{x_0}[\tau^\epsilon(x_0)]$ and

$$\begin{aligned} \hat{\mu} &= \min_{y \in \partial\Psi} \frac{1}{2} y^T N y, \\ N &= \left(DXD^T\right)^{-1}, AX + XA^T + CC^T = 0. \end{aligned} \tag{6}$$

Constant $\hat{\mu}$ is referred to as the logarithmic residence time of (1) in Ψ.

Consider now the controlled linear system

$$\begin{aligned} dx &= (Ax + Bu)dt + \epsilon C dw \\ y &= Dx \end{aligned} \tag{7}$$

where $u \in R^m$ is the control. We assume that there is available for control purposes a measured output $z \in R^q$ and consider control laws of the form

$$\begin{aligned} u &= K\hat{x} \\ \dot{\hat{x}} &= A\hat{x} + Bu + L(z - E\hat{x}) \end{aligned} \tag{8}$$

if the measured output $z = Ex$ is noise free, or

$$\begin{aligned} u &= K\hat{x} \\ d\hat{x} &= (A\hat{x} + Bu)dt + L(dz - E\hat{x}dt) \end{aligned} \tag{9}$$

if the measured output $dz = Exdt + \epsilon F dv$ is noisy. Here, $v(t)$ is a q-dimensional standard Brownian motion and, as before, $0 < \epsilon \ll 1$.

For system (7) with control (8) or (9) we consider the following *residence time control* formulation of the aiming control problem: For a given pair $\{\Psi, \tau\}$ select the pair (K, L) such that the residence time in Ψ, $\bar{\tau}^\epsilon(x_0, u)$, satisfies $\bar{\tau}^\epsilon(x_0, u) \geq \tau$.

Let $\bar{y}(t, x_0, \hat{x}_0, K, L)$ be the solution of the deterministic system

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} &= \begin{bmatrix} A & BK \\ LE & A + BK - LE \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \begin{bmatrix} x(0) \\ \hat{x}(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ \hat{x}_0 \end{bmatrix} \\ y &= Dx. \end{aligned} \tag{10}$$

Define

$$\Omega(K, L) = \left\{\begin{bmatrix} x_0 \\ \hat{x}_0 \end{bmatrix} \in R^{2n} | \bar{y}(t, x_0, \hat{x}_0, K, L) \in \Psi, t \geq 0\right\}. \tag{11}$$

Then with regard to control system (7) with controller (8) or (9), Theorem 1 allows us to conclude that for sufficiently small ϵ and $\begin{bmatrix} x_0 \\ \hat{x}_0 \end{bmatrix} \in \Omega(K, L)$, the above formulation of the residence time control problem can be replaced by the alternative problem of selecting the pair (K, L) such that

$$\hat{\mu}(K, L) > \mu \tag{12}$$

where $\hat{\mu}(K, L)$ is the logarirhmic residence time of the closed loop system (7), (8) or (7), (9) and $\mu = \epsilon^2 \ln \tau$. This is the problem considered in this paper.

Definition:

i. System (7) is said to be weakly residence time controllable if for any bounded $\Psi \subset R^p (0 \in \Psi)$ there exists a controller (8) or (9) such that $\hat{\mu}(K, L) > 0$;
ii. System (7) is said to be strongly residence time controllable if for any bounded $\Psi \subset R^p (0 \in \Psi)$ and $\mu > 0$ there exists a controller (8) or (9) such that $\hat{\mu}(K, L) > \mu$.

Throughout the paper we make the following assumptions:

1. (A, C) is disturbable,
2. (D, A) is detectable,
3. $FF^T > 0$, and $w(t)$ and $v(t)$ are independent Brownian motions.
4. transfer matrices $G_c(s) = D(sI - A)^{-1}B$, $G_w(s) = D(sI - A)^{-1}C$ and $G_{z1}(s) = E(sI - A)^{-1}C$ have full normal rank.

RESIDENCE TIME CONTROLLABILITY

In this section we analyze the achieveable residence time of system (7) with controllers (8) and (9) for the noise-free and noisy measurement cases, respectively.

Let $\mathbf{K} = \{K \in R^{m \times n} | A + BK \text{ is Hurwitz}\}$, $\mathbf{L} = \{L \in R^{n \times q} | A - LE \text{ is Hurwitz}\}$ and define the maximal logarithmic residence time of (7) in Ψ with control (8) or (9) as

$$\hat{\mu}^* = \sup_{K \in \mathbf{K}, L \in \mathbf{L}} \hat{\mu}(K, L). \tag{13}$$

Noise-free measurements

We begin by introducing the following hypothesis:

I. $G_s(s)$ is right invertible and minimum phase.
II. $G_{n1}(s)$ is left invertible and minimum phase.
III. There exists an $m \times r$ rational matrix $U(s)$ with no poles in Re $s > 0$ such that $G_n(s) + G_s(s)U(s) = 0$.
IV. There exists an $p \times q$ rational matrix $V(s)$ with no poles in Re $s > 0$ such that $G_n(s) + V(s)G_{n1}(s) = 0$.

Theorem 2: System (7) is

a. weakly residence time controllable by controller (8) if and only if (A, B) is stabilizable and (E, A) is detectable;
b. strongly residence time controllable by controller (8) if and only if (A, B) is stabilizable, (E, A) is detectable and either I and IV or II and III are true.

Remark 1: In SISO case with $D = E$, Theorem 2 implies that for strong residence time controllability $G_s(s)$ should be minimum phase.

The following results, derived earlier in Meerkov and Runolfsson (1988, 1989), for systems with state feedback control laws, $u = Kx$, can be derived from Theorem 2.

Corollary 1: Assume that $E = I$ (the $n \times n$ identity matrix). Then system (7) is

a. *wrt*-controllable by controller (8) if and only if (A, B) is stabilizable;
b. *srt*-controllable if and only if (A, B) is stabilizable and III is true.

For the special case when the controlled output is the whole state vector we have:

Corollary 2: Assume $D = I$. Then system (7) is *srt*-controllable if and only if (A, B) is stabilizable and Im $C \subseteq$ Im B.

Remark 2: Note that I is a stronger condition than III. Thus, either IV or II is the additional condition that has to be satisfied when state feedback is replaced by output feedback.

Noisy measurements

Theorem 3: Let P be the unique positive definite solution of the Riccati equation

$$AP + PA^T + CC^T - PE^T\left(FF^T\right)^{-1}EP = 0. \qquad (14)$$

Then the maximal logarithmic residence time of the closed loop system (7), (9) in Ψ satisfies

$$\hat{\mu}^* \leq \min_{y \in \partial\Psi} \frac{1}{2} y^T \left(DPD^T\right)^{-1} y. \qquad (15)$$

Remark 3: It follows, in particular from Theorem 3 that since the upper bound in (15) is always finite, system (7) with control (9) is never strongly residence time controllable. Therefore, the measurement noise has a greater limiting effect on the achieveable residence time than the input noise in (7).

The following theorem illustrates that the upper bound in (15) is the best possible bound.

Theorem 4: The upper bound in (15) is attained if and only if there exists a rational matrix $W(s)$ with no poles in Re $s > 0$ such that

$$G_l(s) + G_s(s)W(s) = 0 \qquad (16)$$

where $G_s(s)$ is defined as previously and

$$G_l(s) = D(sI - A)^{-1}\hat{L}, \quad \hat{L} = PE^T\left(FF^T\right)^{-1}. \qquad (17)$$

DESIGN TECHNIQUES

In the last section we characterized the achievable residence time in systems with observer based control laws and noise-free and noisy measurements. In this section we develop controller design techniques that achieve (or approach) the maximal logarithmic residence time. We will concentrate on the noisy measurement case and illustrate how the noise-free case as well as state feedback can be obtained in a similar way.

Assume, for simplicity, that the domain Ψ is an ellipsoid

$$\Psi = \left\{y \in R^p | y^T S y \leq r^2, S = S^T > 0\right\}. \qquad (18)$$

Let $W \in R^{p \times p}$ be a nonsingular matrix such that $S = WW^T$. Then a straight forward calculation gives

$$\hat{\mu}(K, L) = \frac{r^2}{2\lambda_{max}\left[WDX(K,L)D^TW^T\right]} \qquad (19)$$

where $\lambda_{max}[\cdot]$ denotes the maximum eigenvalue of a symmetric matrix and $X(K, L)$ is defined by

$$\begin{bmatrix} A & BK \\ LE & A + BK - LE \end{bmatrix} \begin{bmatrix} X(K,L) & T(K,L) \\ T^T(K,L) & \hat{X}(K,L) \end{bmatrix} + \begin{bmatrix} X(K,L) & T(K,L) \\ T^T(K,L) & \hat{X}(K,L) \end{bmatrix} \begin{bmatrix} A & BK \\ LE & A + BK - LE \end{bmatrix}^T + \begin{bmatrix} CC^T & 0 \\ 0 & LFF^TL^T \end{bmatrix} = 0. \qquad (20)$$

From (19) we conclude that a pair (K, L) is optimal if and only if it minimizes the largest eigenvalue of $\Gamma(K, L) = WDX(K, L)D^TW^T$. The following lemma, whose proof is similar to the proof of Theorem 2.1 in Allwright and Mao (1982), characterizes the minimum value of $\lambda_{max}[\Gamma(K, L)]$.

Lemma 1: Let $\theta \geq 0$ be a scalar, $l \geq 1$ be an integer and select $K_l \in \mathbf{K}$ and $L_l \in \mathbf{L}$ such that

$$\mathrm{Tr}\,\Gamma(K_l, L_l)^l \leq (1 + \theta) \inf_{K \in \mathbf{K}, L \in \mathbf{L}} \mathrm{Tr}\,\Gamma(K, L)^l. \qquad (21)$$

Then

$$\lim_{l \to \infty} \lambda_{max}[\Gamma(K_l, L_l)] = \inf_{K \in \mathbf{K}, L \in \mathbf{L}} \lambda_{max}[\Gamma(K, L)]. \qquad (22)$$

It follows from the lemma that in order to minimize $\lambda_{max}[\Gamma(K,L)]$, it suffices to minmize $\mathrm{Tr}\,\Gamma(K,L)^l$ for $l = 1,2,3,\ldots$. To accomplish this introduce the regularized "cost"

$$J_\gamma^l(K,L) = \mathrm{Tr}\left(\Gamma(K,L)^l + \gamma K\hat{X}(K,L)K^T\right) \tag{23}$$

where $\hat{X}(K,L)$ is given by (20).

Lemma 2: Assume $K_l^\gamma \in \mathbf{K}$ and $L_l^\gamma \in \mathbf{L}$ minimize $J_\gamma^l(K,L)$. Then

$$\lim_{\gamma\to 0} J_\gamma^l(K_l^\gamma, L_l^\gamma) = \inf_{K\in\mathbf{K}, L\in\mathbf{L}} \mathrm{Tr}\,\Gamma(K,L)^l. \tag{24}$$

From Lemmas 1 and 2 we obtain

Corollary 3: Assume that the pair $(K_l^\gamma, L_l^\gamma) \in \mathbf{K}\times\mathbf{L}$ minmizes $J_\gamma^l(K,L)$. Then

$$\lim_{l\to\infty}\lim_{\gamma\to 0}\hat{\mu}(K_l^\gamma, L_l^\gamma) = \hat{\mu}^*. \tag{25}$$

A necessary condition for the optimality of (K_l^γ, L_l^γ) in the sense of minmizing functional (23) is given in the following theorem.

Theorem 5: Assume that $(K_l^\gamma, L_l^\gamma) \in \mathbf{K}\times\mathbf{L}$. Then in order for (K_l^γ, L_l^γ) to minimize $J_\gamma^l(K,L)$ it is necessary that

$$L_l^\gamma = \hat{L} = PE^T\left(FF^T\right)^{-1}, \tag{26}$$

$$K_l^\gamma = -\frac{1}{\gamma}B^T Q_l^\gamma \tag{27}$$

where P is given by (14) and

$$A^T Q_l^\gamma + Q_l^\gamma A + D^T W^T M_l^\gamma W D - \frac{1}{\gamma}Q_l^\gamma B B^T Q_l^\gamma = 0, \tag{28}$$

$$M_l^\gamma = l\left(WD\left(\hat{X}_l^\gamma + P\right)D^T W^T\right)^{l-1}, \tag{29}$$

$$(A + BK_l^\gamma)\hat{X}_l^\gamma + \hat{X}_l^\gamma(A + BK_l^\gamma)^T + \hat{L}\hat{L}^T = 0. \tag{30}$$

Since (14) has a positive solution, $L_l^\gamma = \hat{L} \in \mathbf{L}, \forall\gamma, l$. The following lemma gives a condition for $K_l^\gamma \in \mathbf{K}$.

Lemmas 3: Assume that $M_l^\gamma > 0$. Then $K_l^\gamma \in \mathbf{K}$.

Remark 4: It follows from Theorem 5 that the optimal observation gain is independent of the optimal control whereas the optimal control gain depends on the optimal observations. Thus, a semi-separation principle holds.

Remark 5: The optimal estimator (observer) gain $\hat{L}$ is the Kalman filter gain. Thus, the Kalman filter is optimal for optimization problem (13). Furthermore, consider the equation for the estimation error $e = x - \hat{x}$

$$de = (A - LE)e\,dt + \epsilon(C\,dw - LF\,dv) \tag{31}$$

and denote the logarithmic residence time of e in a bounded domain $\Upsilon \subset R^n\,(0 \in \Upsilon)$ by $\hat{\mu}_\Upsilon(L)$. Then

$$\hat{\mu}_\Upsilon(L) = \min_{e\in\partial\Upsilon}\frac{1}{2}e^T P^{-1}(L)e \tag{32}$$

where $P(L)$ is the positive definite solution of

$$(A - LE)P(L) + P(L)(A - LE)^T + CC^T + LFF^T L^T = 0. \tag{33}$$

Since P given by (14) satisfies

$$P \le P(L), \forall L \in \mathbf{L}, \tag{34}$$

we conclude that

$$\hat{\mu}_\Upsilon\left(\hat{L}\right) = \min_{e\in\partial\Upsilon}\frac{1}{2}e^T P^{-1}e \ge \hat{\mu}_\Upsilon(L), \forall L \in \mathbf{L}. \tag{35}$$

Therefore, the Kalman filter is optimal in the sense of maximizing the logarithmic estimation error residence time in any bounded domain $\Upsilon \subset R^n\,(0 \in \Upsilon)$.

Remark 6: The optimal control, law for system (7) with noise free measurements and control law (8) can be obtained from (26) — (27) by selecting $F = \alpha I$ and letting $\alpha \to 0$. Indeed, since the optimal estimator law for (7), (9) is the Kalman filter we know from the theory of optimal filtering that the (singular) optimal filter for (7), (8) is obtained in the limit $\alpha \to 0$ (see, e.g., Kwakernaak and Sivan (1972)). Therefore, the maximal logarithmic residence time for (7), (8) is

$$\hat{\mu}^* = \lim_{l\to\infty}\lim_{\gamma\to 0}\lim_{\alpha\to 0}\hat{\mu}(K_l^{\gamma,\alpha}, L_l^{\gamma,\alpha}) \tag{36}$$

where $K_l^{\gamma,\alpha}$ and $L_l^{\gamma,\alpha}$ are given by (26) — (30) with $FF^T = \alpha^2 I$.

Remark 7: The state feedback controller that maximizes the logarithmic residence time of (7) in Ψ can be consructed in a similar way as the optimal controller in Theorem 5. In particular, in this case we obtain that the controller

$$u = \tilde{K}_l^\gamma x, \tag{37}$$

$$\tilde{K}_l^\gamma = -\frac{1}{\gamma}B^T\tilde{Q}_l^\gamma, \tag{38}$$

$$A^T\tilde{Q}_l^\gamma + \tilde{Q}_l^\gamma A + D^T W^T \tilde{M}_l^\gamma W D - \frac{1}{\gamma}\tilde{Q}_l^\gamma B B^T \tilde{Q}_l^\gamma = 0, \tag{39}$$

$$\tilde{M}_l^\gamma = l\left(WDX_l^\gamma D^T W^T\right)^{l-1}, \tag{40}$$

$$\left(A + B\tilde{K}_l^\gamma\right)X_l^\gamma + X_l^\gamma\left(A + B\tilde{K}_l^\gamma\right)^T + CC^T = 0 \tag{41}$$

results in a logarithmic residence time, $\hat{\mu}\left(\tilde{K}_l^\gamma\right)$, that satisfies

$$\lim_{l\to\infty}\lim_{\gamma\to 0}\hat{\mu}\left(\tilde{K}_l^\gamma\right) = \sup_{K\in\mathbf{K}}\hat{\mu}(K). \tag{42}$$

EXAMPLE

Consider the second order system

$$\begin{aligned}\dot{x} &= \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}x + \begin{bmatrix}0\\1\end{bmatrix}u + \epsilon\begin{bmatrix}1\\0\end{bmatrix}\dot{w},\\ y &= [0 \quad 1]x,\\ z &= [1 \quad 0]x + \epsilon F\dot{v}.\end{aligned} \tag{43}$$

For this system

$$
\begin{aligned}
G_s(s) &= \frac{s}{s^2+1}, \\
G_n(s) &= \frac{-1}{s^2+1}, \\
G_{n1}(s) &= \frac{s}{s^2+1}.
\end{aligned}
\tag{44}
$$

Therefore, since $G_s(s) \equiv G_{n1}(s)$ is minimum phase, the system is art-controllable by controller (8) when $F \equiv 0$.

Assume $F \neq 0$. Then, by Theorem 3, the logarithmic residence time in the interval $\Psi = (-a, b)$, $a, b > 0$, is bounded by

$$\min_{y \in \partial \Psi} \frac{1}{2} y^T \left(DPD^T\right)^{-1} y = \frac{(\min(a,b))^2}{2|F|}. \tag{45}$$

Furthermore, when $a = b$, the optimal controller given in Theorem 5 is

$$
\begin{aligned}
K_l^\gamma &= -[0 \quad K_2], \\
\hat{L} &= \begin{bmatrix} \frac{1}{|F|} \\ 0 \end{bmatrix}
\end{aligned}
\tag{46}
$$

where $K_2 > 0$ satisfies the equation

$$\frac{K_2^2 \gamma}{l|F|^{l-1}} = \left(1 + \frac{|F|}{2K_2}\right)^{l-1}. \tag{47}$$

The logarithmic residence time with this control is

$$\hat{\mu}\left(K_l^\gamma, \hat{L}\right) = \frac{a^2}{2|F|} \cdot \frac{2K_2}{2K_2 + |F|}. \tag{48}$$

Note that $\hat{\mu}\left(K_l^\gamma, \hat{L}\right)$ is the upper bound in (39) multilpied by the factor

$$\rho = \frac{2K_2}{2K_2 + |F|}. \tag{49}$$

Note that $\rho \to 1$ as $\gamma \to 0$ and $l \to \infty$.

In order to obtain a logarithmic residence time as close as desired to the maximal value (39), (43) can be used to calculate the necessary K_2 (for a given ρ) and l and γ can be determined from(41).

As $\gamma \to 0$ equation (41) simplifies considerably. Indeed, in this case $K_2 \to \infty$ and, thus, for small γ (41) becomes

$$\frac{K_2^2 \gamma}{l|F|^{l-1}} \approx 1. \tag{50}$$

Finally, this gives

$$K_2 \approx \sqrt{\frac{l}{\gamma}} |F|^{\frac{l-1}{2}}. \tag{51}$$

CONCLUSIONS

It is shown in this paper that the fundamental bounds on the achievable precision of aiming of linear systems perturbed by white noise depend on the locations of nonminimum phase zeros of the various transfer functions involved, and on the dimensions of the controlled and measured outputs and control and noise inputs. Roughly speaking, the best precision of aiming is obtained for minimum phase systems with the number of control inputs larger than outputs. Any desired residence time is attainable only if no measurement noise is present. Therefore, the effect of measurement noise is more detrimental on the precision of aiming than that of the input system noise.

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APPENDIX 2

Output Residence Time Control

S. M. MEERKOV AND T. RUNOLFSSON

Abstract—**The problem of residence time control, introduced in [1], is extended to systems with outputs. Necessary and sufficient conditions for output residence time controllability in linear systems with small, additive noise are derived. State feedback controller design techniques are developed and applied to a robotics control problem. The approach is based on an extension of the asymptotic first passage time theory to output processes.**

I. Introduction

Given a controlled dynamical system with states $x(t) \in \mathbb{R}^n$, control $u(t) \in \mathbb{R}^m$, output $y(t) \in \mathbb{R}^p$, and disturbances $\xi(t) \in \mathbb{R}^r$, assume its desired behavior is specified by a pair $\{\Psi, \tau\}$, where $\Psi \subset \mathbb{R}^p$ is the domain to which the outputs $y(t)$ should be confined and τ is the period of the confinement, i.e., $y(t) \in \Psi$, $\forall t \in [t_0, t_0 + \tau]$, $t_0 \in \mathbb{R}_+$. Problem formulation of this form arises in numerous applications. For instance, in the problem of telescope pointing, the domain Ψ is defined by the film grain size and τ is the exposure time (see [1] for additional examples).

For a given pair $\{\Psi, \tau\}$, the problem of *output residence time control* is formulated as the problem of choosing a feedback control law, so as to force $y(t)$ to remain, at least on average, in Ψ during period τ, in spite of the disturbances $\xi(t)$ that are acting on the system.

The purpose of the present note is to analyze the fundamental capabilities and limitations of *output* residence time control for linear systems with small additive perturbations. The approach is based on an extension of the asymptotic first passage time theory to output processes.

The structure of the note is as follows: in Section II the notion of an output residence time is introduced; in Section III output residence time controllability is defined and analyzed; in Section IV output residence time controller techniques are given; and in Section V an example is considered. The proofs are given in the Appendix.

II. Output Residence Time

Consider a linear stochastic system

$$dx = Axdt + \epsilon Cdw$$

$$y = Dx \tag{2.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $w(t)$ is a standard r-dimensional Brownian motion and $0 < \epsilon \ll 1$ is a parameter. It is assumed, without loss of generality, the rank $D = p$.

Let $\Psi \subset \mathbb{R}^p$ be an open bounded domain containing the origin and whose boundary $\partial\Psi$ is smooth and define

$$\Omega_0 = \{x \in \mathbb{R}^n | y = Dx \in \Psi\},$$

$$\Omega = \{x \in \mathbb{R}^n | De^{At}x \in \Psi, t \geq 0\}.$$

Assume that $x_0 = x(0) \in \Omega_0$ and denote as $y(t, x_0)$ the output $y(t)$ defined by (2.1) with the initial condition x_0. Introduce the first passage time of $y(t, x_0)$ from Ψ as follows:

$$\tau^\epsilon(x_0) = \inf\{t \geq 0 : y(t, x_0) \in \partial\Psi\} \tag{2.2}$$

and its mean

$$\bar{\tau}^\epsilon(x_0) = E\{\tau^\epsilon(x_0) | x_0\}. \tag{2.3}$$

The calculation of $\bar{\tau}^\epsilon(x_0)$ is, in general, a difficult task. To alleviate this difficulty, asymptotic approximations with respect to small ϵ can be used. For the special case $y(t) \equiv x(t)$ these approximations have been extensively discussed in the literature (see, e.g., [1]-[4] and references

therein). An extension to the more general case of $y(t) = Dx(t)$ is given below (see also [4]).

Theorem 2.1: Assume that A is Hurwitz and (A, C) is completely disturbable, i.e., rank $[C\ AC \cdots A^{n-1}C] = n$. Then uniformly for all x_0 belonging to compact subsets of Ω we have

$$\lim_{\epsilon \to 0} \epsilon^2 \ln \bar{t}^{\epsilon}(x_0) = \hat{\mu}(\Psi) \tag{2.4}$$

where

$$\hat{\mu}(\Psi) = \min_{y \in \partial\Psi} \frac{1}{2} y^T N y, \qquad N = (DXD^T)^{-1} \tag{2.5}$$

and X is the positive definite solution of

$$AX + XA^T + CC^T = 0. \tag{2.6}$$

Proof: See the Appendix.

The constant $\hat{\mu}(\Psi)$ is referred to as the *logarithmic residence time* in Ψ. The properties of this constant, as stated in Theorem 2.1, constitute the mathematical foundation for the analysis in Sections III and IV.

If y is a scalar, the logarithmic residence time can be expressed in a more traditional form. Indeed, since in this case Ψ is an interval, say, $\Psi = (-a, b)$, $a, b > 0$, and N is a scalar, from (2.5) it follows that

$$\hat{\mu}(\Psi) = \frac{1}{2} (\min (a, b))^2 N,$$

$$N = \left(\int_0^\infty De^{At}CC^Te^{A^Tt}D^T\, dt \right)^{-1} = \left(\int_0^\infty \mathrm{Tr}\, C^Te^{A^Tt}D^TDe^{At}C\, dt \right)^{-1}$$

$$= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |G_n(j\omega)|^2\, dw \right)^{-1} = \|G_n\|_2^{-2}, \quad G_n(s) = D(sI - A)^{-1}C.$$

Therefore,

$$\hat{\mu}(\Psi) = \frac{\min (a, b))^2}{2\|G_n\|_2^2}. \tag{2.7}$$

When y is not a scalar, the simple relationship (2.7) is not true. In fact, it is not difficult to show that in general

$$\hat{\mu}(\Psi) \geq \frac{\min_{y \in \partial\Psi} y^Ty}{2\|G_n\|_2^2}. \tag{2.8}$$

III. Output Residence Time Controllability

Consider now a controlled linear stochastic system

$$dx = (Ax + Bu)dt + \epsilon C dw, \qquad u \in \mathbb{R}^m$$

$$y = Dx. \tag{3.1}$$

Let $u = Kx$ and let $\bar{t}^{\epsilon}(x_0, K)$ be the mean first passage time from Ψ of the closed-loop system

$$dx = (A + BK)x dt + \epsilon C dw$$

$$y = Dx \tag{3.2}$$

with initial conditions $x_0 = x(0) \in \Omega_K = \{x \in \mathbb{R}^n : De^{(A+BK)T}x \in \Psi, T \geq 0\}$. Define

$$\hat{\mu}(\Psi, K) = \lim_{\epsilon \to 0} \epsilon^2 \ln \bar{t}^{\epsilon}(x_0, K). \tag{3.3}$$

Definition:

i) The output $y(t)$ of system (3.1) is said to be *weakly residence time controllable* (y-wrt controllable) if for any bounded domain $\Psi \subset \mathbb{R}^p$ with 0 in its interior, there exists a control $u = Kx$ such that $\hat{\mu}(\Psi, K) > 0$.

ii) $y(t)$ is said to be *strongly residence time controllable* (y-srt

of

controllable) if for any bounded $\Psi \subset \mathbb{R}^p$ ($0 \in \Psi$) and $\mu > 0$ there exists $u = Kx$ such that $\hat{\mu}(\Psi, K) \geq \mu$.

The following theorem characterizes the class of y-wrt controllable systems.

Theorem 3.1: Assume that (3.1) has no modes that are both uncontrollable and undisturbable. Then system (3.1) is y-wrt controllable if (A, B) is stabilizable. If (D, A) is detectable then stabilizability of (A, B) is also a necessary condition for y-wrt controllability.

Proof: See the Appendix.

The assumption that (3.1) has no modes that are both uncontrollable controllable and undisturbable, i.e., $(A, [B\ C])$ is a ~~controller~~ pair, is assumed to be true in the remainder of the note. It is made to rule out some mathematical degeneracies. Methods for relaxing this assumption are discussed in [1].

Next we give conditions for y-srt controllability. Define transfer matrices

$$G_r(s) = D(sI - A)^{-1}B \tag{3.4}$$

and

$$G_n(s) = D(sI - A)^{-1}C. \tag{3.5}$$

Theorem 3.2: Assume (A, B) is stabilizable and (D, A) is detectable. Then (3.1) is y-srt controllable if and only if there exists an $m \times r$ rational matrix $U(s)$ with no poles in the open right-half complex plane such that

$$G_n(s) + G_r(s)U(s) = 0. \tag{3.6}$$

Proof: See the Appendix.

Remark 3.1: If $G_n(s)$ and $G_r(s)$ are scalars then it follows from Theorem 3.2 that (3.1) is y-srt controllable if and only if all nonminimum phase zeros of $G_r(s)$ are also zeros of $G_n(s)$.

Remark 3.2: When D is an $n \times n$ nonsingular matrix, i.e., when (3.1) has as many outputs as states, condition (3.6) becomes $\mathrm{Im} C \subseteq \mathrm{Im} B$. This condition has been earlier derived in [1] (see also [4]).

Although the above results are formulated in terms of multivariable systems, to simplify the situation, in the remainder of the note we assume that y, u, and w are scalars and address the following problems: (:)

Problem 1: What is the fundamental bound on the achievable logarithmic residence time of an output which is not y-srt controllable?

Problem 2: How to design a controller which results in a desired output logarithmic residence time?

¶ We give the solution to Problem 1 in this section and to Problem 2 in Section IV.

¶ Let $K = \{K \mid A + BK \text{ is Hurwitz}\}$ and define the ***maximal logarithmic residence time*** in Ψ by

K bold

$$\mu^*(\Psi) = \sup_{K \in \mathcal{K}} \hat{\mu}(\Psi, K). \tag{3.7}$$

bold

Obviously, $\mu^*(\Psi) = \infty$ for a y-srt controllable output. Let $z_1, \cdots, z_l$ be the open right-half plane (rhp) zeros of $G_r(s)$.

Theorem 3.3: Assume (A, B) is stabilizable. Then $\mu^*(\Psi)$ is given by

$$\mu^*(\Psi) = \frac{(\min(a, b))^2}{2\|G_0\|_2^2}, \tag{3.8}$$

$$G_0(s) = \frac{q(s)}{\prod_{i=1}^{l} (\bar{z}_i + s)} \tag{3.9}$$

where $\bar{z}_i$ is the complex conjugate of z_i and $q(s)$ is the unique polynomial of degree less than l determined by the interpolation constraints: at each rph zero z of $G_r(s)$ of multiplicity m, $G_0(s)$ satisfies

$$\frac{d^k}{ds^k} G_0(s)\Big|_{s=z} = \frac{d^k}{ds^k} G_n(s)\Big|_{s=z}, \quad k = 0, \cdots, m-1. \tag{3.10}$$

Proof: See the Appendix.

Remark 3.3: The function $G_0(s)$ defined by (3.9) is the rational function of minimum H^2-norm which satisfies the interpolation con-

straints (3.10) [5], [6]. Thus, the problem (3.7) is equivalent to the problem

$$\min \{\|G\|_2 : G(s) \in H^2,\ G(s) \text{ rational}\}$$

subject to the constraints (3.10). On the other hand, the H^∞-optimal transfer function [7] which satisfies the interpolation constraints (3.10) is an all pass filter, i.e., constant in magnitude on the $j\omega$-axis. Thus, an unweighted H^∞-optimal, stabilizing state feedback controller [8] leads to a closed-loop system with the shortest possible logarithmic residence time.

The formula (3.8) for $\mu^*(\Psi)$ simplifies considerably when the rhp zeros of $G_s(s)$ are distinct. Define an $l \times l$ matrix Z by $z_{ij} = (z_i + \bar{z}_j)^{-1}$, $1 \leq i, j \leq l$, and an $l \times 1$ column vector g by $g_j = G_a(z_j)$, $j = 1, \cdots, l$.

1 *Theorem 3.4:* Assume that the rhp zeros of $G_s(s)$ are all distinct. Then

$$\mu^*(\Psi) = \frac{(\min(a, b))^2}{2g^H Z^{-1} g} \tag{3.11}$$

where g^H is the Hermitian transpose of g, i.e., $g^H = \bar{g}^T$.

Proof: See Appendix

IV. Output Residence Time Controller Design

In this section we give a method for selecting a controller which results in any admissible logarithmic residence time $\hat{\mu} < \mu^*(\Psi)$.

Recall that $\hat{\mu}(\Psi, K)$ is given by

$$\hat{\mu}(\Psi, K) = \frac{1}{2}(\min(a, b))^2 N(K). \tag{4.1}$$

Thus, the maximization problem (3.7) is equivalent to the minimization problem

$$\inf_{K \in \mathcal{K}} DX(K)D^T. \tag{4.2}$$

Since the equation which $X(K)$ satisfies is linear in K, it is easy to see that the infimum (4.2) is not attained at any $K \in \mathcal{K}$. Thus, $\mu^*(\Psi)$ is not attained for any $K \in \mathcal{K}$. We now construct a sequence of controllers whose logarithmic residence times ~~coverage~~ converge to $\mu^*(\Psi)$. Let $K_0 \in \mathcal{K}$ and define a regularized "cost"

$$J_\gamma(K) = DX(K_0 + K)D^T + \gamma KX(K_0 + K)K^T, \qquad \gamma > 0. \tag{4.3}$$

Obviously,

$$\hat{\mu}(\Psi, K_0 + K) \geq \frac{(\min(a, b))^2}{2J_\gamma(K)}, \qquad K \in \mathcal{K}. \tag{4.4}$$

It is well known from the theory of optimal control [9] that $J_\gamma(K)$ is minimized by

$$K^\gamma = -\frac{1}{\gamma} B^T Q_\gamma \tag{4.5}$$

where Q_γ is the positive semidefinite solution of

$$(A + BK_0)^T Q_\gamma + Q_\gamma (A + BK_0) + D^T D - \frac{1}{\gamma} Q_\gamma B B^T Q_\gamma = 0. \tag{4.6}$$

The following theorem can be proved using the results of [9].

Theorem 4.1: Assume (A, B) is stabilizable. Then $\hat{\mu}(\Psi, K_0 + K^\gamma)$ is nondecreasing as $\gamma \to 0$, and

$$\lim_{\gamma \to 0} \hat{\mu}(\Psi, K_0 + K^\gamma) = \mu^*(\Psi).$$

V. Example

Example 5.1: Consider the problem of controlling the tip position of a flexible robot arm using control torques applied at the robot arms hub

[10]. A finite-dimensional approximate model for a robot arm which is flexible in the horizontal plane but not in the vertical plane or in torsion was described in [10]. The model is described by the following set of equations:

$$\dot{x}_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta_i\omega_i \end{bmatrix} x_i + \begin{bmatrix} 0 \\ \dfrac{1}{I_T}\dfrac{d\phi_i}{dz}(0) \end{bmatrix} u + \begin{bmatrix} \phi_i(L) \\ 0 \end{bmatrix} \xi$$

$$y_i = [\phi_i(L)\ \ 0]x_i, \qquad i = 0, 1, \cdots, n \tag{5.1}$$

where L is the length of the arm, ζ_i, ω_i, and $\phi_i(z)$ ($z \in [0, L]$) are the damping coefficients, pinned-free frequency, and modal gain, respectively, of the ith mode of oscillation, I_T is the total moment of inertia, u is the control torque, ζ is a random torque acting on the tip, and $y = \Sigma_{i=0}^{n} y_i$ is the tip position.

Assume that it is desired to maintain the tip position within the bounds $-a \le y \le b$ during a specified time-interval, T, and assume that the disturbance ξ can be modeled as a small white noise ϵw.

The system transfer function for (5.1) is

$$G_s(s) = \frac{1}{I_T}\sum_{i=0}^{n} \frac{\phi_i(L)\dfrac{d\phi_i(0)}{dz}}{s^2 + 2\zeta_i\omega_i s + \omega_i^2} \tag{5.2}$$

and the noise transfer function is

$$G_n(s) = \sum_{i=0}^{n} \frac{\phi_i^2(L)(s + 2\zeta_i\omega_i)}{s^2 + 2\zeta_i\omega_i s + \omega_i^2}. \tag{5.3}$$

It was indicated in [10] that taking $n = 3$ gives a good approximate model and the values of the constants ζ_i, ω_i, $\phi_i(L)$, $d\,\phi_i(0)/dz$ and I_T were determined experimentally. The resulting system has three right half-plane zeros at $z_1 = 12.04$ and $z_{2,3} = 21.5 \pm j\,25.3$. It is easily checked that $G_n(z_1) \neq 0$. Thus, system (5.1) is not y-srt controllable. However, it is controllable and, thus y-wrt controllable and the maximal logarithmic residence time can be obtained from (3.11) to be

$$\mu^*(\Psi) = \frac{(\min(a, b))^2}{2g^H Z^{-1} g} = \frac{(\min(a, b))^2}{2(0.046)} = 10.87\,(\min(a, b))^2. \tag{5.4}$$

Thus, any specified time-interval $[0, T]$ has to satisfy the bound

$$\ln T \le \frac{10.87 \min(a, b))^2}{\epsilon^2}. \tag{5.5}$$

APPENDIX

Proof of Theorem 2.1: First note that it follows from the definition of Ω_0 that

$$\inf\{t \ge 0 | y(t) \in \partial\Psi\} = \inf\{t \ge 0 | x(t) \in \partial\Omega_0\}. \tag{A.1}$$

Therefore,

$$\begin{aligned} t^\epsilon(x_0, \Psi) &= E\,[\inf\{t \ge 0 | y(t, x_0) \in \partial\Psi\} | x_0] \\ &= E\,[\inf\{t \ge 0 | x(t, x_0) \in \partial\Omega_0 | x_0] = t^\epsilon(x_0, \Omega_0). \end{aligned} \tag{A.2}$$

Now, it follows from [4, Theorem 4] that uniformly for x_0 belonging to compact subsets of Ω we have

$$\lim_{\epsilon \to 0} \epsilon^2 \ln t^\epsilon(x_0, \Omega_0) = \hat{\phi}(\Omega_0) = \min_{x \in \partial\Omega_0} \frac{1}{2} x^T M x \tag{A.3}$$

where $M = X^{-1}$ and X is given by (2.6).

To complete the proof we show that $\hat{\phi}(\Omega_0) = \hat{\mu}(\Psi)$. Let $x = T\tilde{x}$ be a nonsingular change of coordinates that maps (2.1) into the form

$$d\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = T^{-1}AT\tilde{x}dt + \epsilon T^{-1}Cdw$$

$$= \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} dt + \epsilon \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix} dw$$

$$y = DT\tilde{x} = \tilde{D}\tilde{x} = [I \ 0] \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \tag{A.4}$$

(such a T always exists since rank $D = p$). Under this change of coordinates M is mapped into $\tilde{M} = T^T MT$ and the domains Ω_0 and Ω become

$$\tilde{\Omega}_0 = \{\tilde{x} \in R^n | \tilde{x}_1 \in \Psi\},$$

$$\tilde{\Omega} = \{\tilde{x}_0 \in R^n | y = \tilde{D}e^{\tilde{A}t}\tilde{x}_0 \in \Psi,\ t \geq 0\}. \tag{A.5}$$

The logarithmic residence time in $\tilde{\Omega}_0$ is now given by

$$\hat{\phi}(\tilde{\Omega}_0) = \min_{\tilde{x} \in \partial\tilde{\Omega}_0} \frac{1}{2} \tilde{x}^T \tilde{M} \tilde{x}$$

$$= \min_{\substack{\tilde{x}_1 \in \partial\Psi \\ \tilde{x}_2 \text{ free}}} \frac{1}{2} [\tilde{x}_1^T \ \tilde{x}_2^T] \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{12}^T & \tilde{M}_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}. \tag{A.6}$$

We minimize first with respect to the unconstrained variables $\tilde{x}_2$ giving

$$\tilde{x}_2 = -\tilde{M}_{22}^{-1} \tilde{M}_{12}^T \tilde{x}_1. \tag{A.7}$$

Substituting (A.7) into (A.6) and rearranging gives

$$\hat{\phi}(\tilde{\Omega}_0) = \min_{\tilde{x}_1 \in \partial\Psi} \frac{1}{2} \tilde{x}_1 [\tilde{M}_{11} - \tilde{M}_{12} \tilde{M}_{22}^{-1} \tilde{M}_{12}^T] \tilde{x}_1. \tag{A.8}$$

The matrix $\tilde{M}_{11} - \tilde{M}_{12} \tilde{M}_{22}^{-1} \tilde{M}_{12}^T$ is exactly $\tilde{X}_{11}^{-1}$ where net itc(: X_{11}

$$\tilde{M}^{-1} = \tilde{X} = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{12}^T & \tilde{X}_{22} \end{bmatrix}.$$

Therefore,

$$\hat{\phi}(\tilde{\Omega}_0) = \min_{\tilde{x}_1 \in \partial\Psi} \frac{1}{2} \tilde{x}_1^T \tilde{X}_{11}^{-1} \tilde{x}_1. \tag{A.9}$$

However, $\tilde{X}_{11} = \tilde{D}\tilde{X}\tilde{D}^T$ and therefore

$$\hat{\phi}(\tilde{\Omega}_0) = \min_{\tilde{x}_1 \in \partial\Psi} \frac{1}{2} \tilde{x}_1^T (\tilde{D}\tilde{X}\tilde{D}^T)^{-1} \tilde{x}_1. \tag{A.10}$$

Finally, substituting back the original coordinates gives $\tilde{D}\tilde{X}\tilde{D}^T = DTT^{-1} X (T^T)^{-1} T^T D^T = DXD^T$. Thus

$$\hat{\phi}(\tilde{\Omega}_0) = \hat{\mu}(\Psi) = \min_{y \in \partial\Psi} \frac{1}{2} y^T N y. \tag{A.11}$$

Q.E.D.

Proof of Theorem 3.1: The sufficiency part of the theorem follows directly from [1, Theorem 3.1].

To prove the necessity note that y-wrt controllability implies that there exists a control $u = Kx$ such that $DX(K)D^T > 0$. Assume, without loss of generality, that the closed-loop system has the Kalman canonic form, i.e.,

$$A + BK = \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & 0 & \tilde{A}_{13} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} \\ 0 & 0 & \tilde{A}_{33} & 0 \\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} \end{bmatrix}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}, \quad D = [D_1 \ 0 \ D_3 \ 0]. \tag{A.12}$$

The subsystem

$$\left(\begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\right)$$

is controllable and the detectability of (D, A) implies that $\tilde{A}_{44}$ is Hurwitz. If we can show that $\tilde{A}_{33}$ is Hurwitz, the proof is complete.

Note that

$$DX(K)D^T = [D_1 \; D_3] \begin{bmatrix} X_{11} & X_{13} \\ X_{13}^T & X_{33} \end{bmatrix} \begin{bmatrix} D_1^T \\ D_3^T \end{bmatrix} = \hat{D}\hat{X}\hat{D}^T \quad \text{(A.13)}$$

where X_{ij}, $1 \le i, j \le 4$ is a decomposition of $X(K)$ compatible with (A.12). Also, $\hat{X}$ satisfies the Lyapunov equation

$$\hat{A}\hat{X} + \hat{X}\hat{A}^T + \hat{C}\hat{C}^T = 0 \quad \text{(A.14)}$$

where

$$\hat{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{13} \\ 0 & \tilde{A}_{33} \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} C_1 \\ C_3 \end{bmatrix}.$$

The pair $(\hat{D}, \hat{A})$ is observable. Therefore, by [11, Corollary 1] all eigenvalues of $\hat{A}$ in $\mathrm{Re}\, s \ge 0$ are undisturbable. However, since $[A, [B \; C]]$ is controllable, we can assume that $(A + BK, C)$ is a disturbable pair (otherwise an arbitrarily small change in K, say δK, will render $(A + B(K + \delta K), C)$ disturbable [12]). Therefore, $(\hat{A}, \hat{C})$ is a disturbable pair and, thus, $\hat{A}$ is Hurwitz. Q.E.D.

Proof of Theorem 3.2: The proof is a simple application of the results in [9] and [13]. It is easy to show that

$$\frac{\min_{y \in \partial\Psi} y^T y}{2 \,\mathrm{Tr}\, DX(K)D^T} \le \hat{\mu}(\Psi, K) \le \frac{p \min_{y \in \partial\Psi} y^T y}{2 \,\mathrm{Tr}\, DX(K)D^T}. \quad \text{(A.15)}$$

mq?

Thus, y-srt controllability is equivalent to

$$\inf_{K \in \mathcal{K}} \mathrm{Tr}\, DX(K)D^T = 0. \quad \text{(A.16)}$$

It follows from the results of [9] that

$$\inf_{K \in \mathcal{K}} \mathrm{Tr}\, DX(K)D^T = \lim_{\gamma \to 0} \mathrm{Tr}\, DX(K^\gamma)D^T \quad \text{(A.17)}$$

where

$$K^\gamma = -\frac{1}{\gamma} B^T Q_\gamma$$

$$A^T Q_\gamma + Q_\gamma A + D^T D - \frac{1}{\gamma} Q_\gamma B B^T Q_\gamma = 0.$$

Furthermore,

$$\lim_{\gamma \to 0} \mathrm{Tr}\, DX(K^\gamma)D^T = \lim_{\gamma \to 0} \mathrm{Tr}\, C^T Q_\gamma C. \quad \text{(A.18)}$$

Thus, it follows from (A.17) and (A.18) that (3.1) is y-srt controllable if and only if

$$\lim_{\gamma \to 0} c_i^T Q_\gamma c_i = 0, \quad i = 1, \cdots, r \quad \text{(A.19)}$$

l.c.

where $C = [c_1 \cdots c_r]$. Now by [13, Theorem 1], (A.19) is true if and only if there exist rational m-vectors $u_i(s)$, $i = 1, \cdots, r$, with no poles in $\mathrm{Re}\, s > 0$ such that

$$D(sI - A)^{-1}[Bu_i(s) + c_i] = 0, \quad i = 1, \cdots, r$$

or equivalently

$$G_1(s)U(s) + G_0(s) = 0 \quad \text{(A.20)}$$

where $U(s) = [u_1(s) \cdots u_r(s)]$. Q.E.D.

Proof of Theorem 3.3: We know from (2.7) and Theorem 4.1 that

$$\mu^*(\Psi) = \lim_{\gamma \to 0} \frac{(\min(a, b))^2}{2\|G_\gamma\|_2^2} \quad \text{(A.21)}$$

where

$$G_\gamma(s) = G_a(s, \hat{K}_\gamma) = D(sI - A - B\hat{K}_\gamma)^{-1}C, \qquad \hat{K}_\gamma = K_0 + K^\gamma. \quad (A.22)$$

Note that $G_\gamma(s)$ can be rewritten as

$$\begin{aligned} G_\gamma(s) &= D(sI - A - B\hat{K}_\gamma)^{-1}C \\ &= D[I - (sI - A)^{-1}B\hat{K}_\gamma]^{-1}(sI - A)^{-1}C \\ &= D[I + (sI - A - B\hat{K}_\gamma)^{-1}B\hat{K}_\gamma](sI - A)^{-1}C \\ &= D(sI - A)^{-1}C + D(sI - A - B\hat{K}_\gamma)^{-1}B\hat{K}_\gamma(sI - A)^{-1}C. \quad (A.23) \end{aligned}$$

Let

$$D(sI - A)^{-1}C = \frac{a(s)}{d(s)} \quad (A.24a)$$

$$D(sI - A - B\hat{K}_\gamma)^{-1}B = \frac{n(s)}{d_\gamma(s)} \quad (A.24b)$$

$$\hat{K}_\gamma(sI - A)^{-1}C = \frac{m_\gamma(s)}{d(s)} \quad (A.24c)$$

and note that $d_\gamma(s) = \det(sI - A - B\hat{K}_\gamma)$ is Hurwitz for all $\gamma > 0$ and $m_\gamma(s) = \hat{K}_\gamma \operatorname{adj}(sI - A)C$ is a polynomial of degree less than n. Then (A.23) becomes

$$G_\gamma(s) = \frac{a(s)d_\gamma(s) + n(s)m_\gamma(s)}{d(s)d_\gamma(s)}. \quad (A.24d)$$

Note that $d_\gamma(s)$ is the denominator polynomial of $G_\gamma(s)$; therefore, $d(s)$ divides $a(s)d_\gamma(s) + n(s)m_\gamma(s)$ for all $\gamma > 0$. Write $n(s) = n_l(s)n_r(s)$ where $n_l(s)$ has zeros only in $\operatorname{Re} s \le 0$ and $n_r(s)$ has zeros in $\operatorname{Re} s > 0$ only. It can be shown that $\sqrt{\gamma}\, d_\gamma(s) \to n_l(s)n_r(-s)$ as $\gamma \to 0$ [14] and $\sqrt{\gamma}\, \hat{K}_\gamma \to \hat{K}$ as $\gamma \to 0$ [9]. Thus,

$$\begin{aligned} G_0(s) = \lim_{\gamma \to 0} G_\gamma(s) &= \frac{a(s)n_l(s)n_r(-s) + n_l(s)n_r(s)m_0(s)}{d(s)n_l(s)n_r(-s)} \\ &= \frac{a(s)n_r(-s) + n_r(s)m_0(s)}{d(s)n_r(-s)} \quad (A.25) \end{aligned}$$

where $m_0(s) = \hat{K} \operatorname{adj}(sI - A)C$. From the previous discussion we know that $a(s)n_r(-s) + n_r(s)m_0(s) = d(s)q(s)$ for some polynomial $q(s)$. Furthermore, since the degrees of $a(s)$ and $m_0(s)$ are less than n and $n_r(s)$ has degree l, it follows that $q(s)$ has degree less than l.

It follows from (A.25) that $G_0(s)$ satisfies the constraints

$$\frac{d^k}{ds^k} G_0(s)|_{s=\xi} = \frac{d^k}{ds^k} G_a(s)|_{s=\xi}, \quad k = 0, \cdots, m-1 \quad (A.26)$$

at each nonminimum phase zero of $G_t(s)$ of multiplicity m. Therefore,

$$G_0(s) = \frac{q(s)}{n_r(-s)} \quad (A.27)$$

and $q(s)$ is uniquely determined by (A.26). Q.E.D.

Proof of Theorem 3.4: By assumption, the nonminimum phase zeros of $G_t(s)$ are distinct. Therefore, we can rewrite $G_0(s)$ as

$$G_0(s) = \sum_{j=1}^{l} \frac{t_j}{\xi_j + s} \quad (A.28)$$

where t_j, $j = 1, \cdots, l$ are some constants. At each ξ_i we have [from (3.10)]

$$G_0(\xi_i) = \sum_{j=1}^{l} \frac{t_j}{\xi_i + \xi_j} = G_a(\xi_i) = g_i. \quad (A.29)$$

Thus, (A.29) gives l equations which can be written in matrix notation as

$$Zt = g \tag{A.30}$$

where $t^T = (t_1, \cdots, t_l)$ and Z and g are defined as previously.

Next we calculate $\|G_0\|_2^2$,

$$\|G_0\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G_0(j\omega)|^2\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{j=1}^{l} \frac{t_j}{\bar{z}_j + j\omega} \sum_{i=1}^{l} \frac{\bar{t}_i}{z_i - j\omega}\, d\omega$$

$$= \sum_{i,j=1}^{l} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{t_j \bar{t}_i}{(\bar{z}_j + j\omega)(z_i - j\omega)}\, d\omega$$

$$= \sum_{i,j=1}^{l} \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{t_j \bar{t}_i}{(\bar{z}_j + s)(z_i - s)}\, ds. \tag{A.31}$$

Using the calculus of residues to evaluate the integrals appearing in (A.31) gives

$$\int_{-j\infty}^{j\infty} \frac{t_j \bar{t}_i}{(\bar{z}_j + s)(z_i - s)}\, ds = 2\pi j\, \frac{t_j \bar{t}_i}{z_i + \bar{z}_j}. \tag{A.32}$$

Thus

$$\|G_0\|_2^2 = \sum_{i,j=1}^{l} \frac{\bar{t}_i t_j}{z_i + \bar{z}_j} = t^H Z t. \tag{A.33}$$

Substituting. $t = Z^{-1} g$ from (A.30) (note that Z is an invertible Hermitian matrix) gives

$$\|G_0\|_2^2 = g^H Z^{-1} g \tag{A.34}$$

and (3.11) follows from (A.34) and (3.8). Q.E.D.

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